itute.com

Online & home tutors Registered business name: itute ABN: 96 297 924 083

2017 Specialist Mathematics

Year 12 Application Task

Time allowed: 4 hours plus

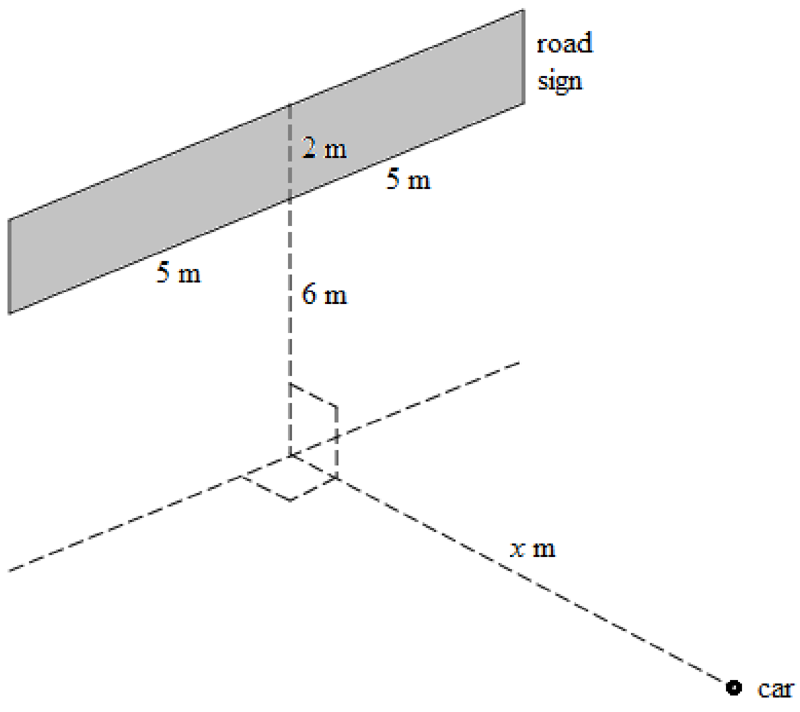
(Assumed knowledge: Pythagoras theorem, 2D and 3D, cosine rule, circular functions, compound and double angle formulas, inverse circular functions, graphs and relations, derivatives, rates of change, related rates, maximum and minimum values, CAS)

You are allowed: 1 bounded reference, 1 CAS, 1 scientific calculator
Working must be shown for questions worth 2 or more marks
Express answers in simplest form
Total 160 marks

Theme: Viewing angle from a distance

Measure angles in radians, and distance in metres unless specified otherwise.

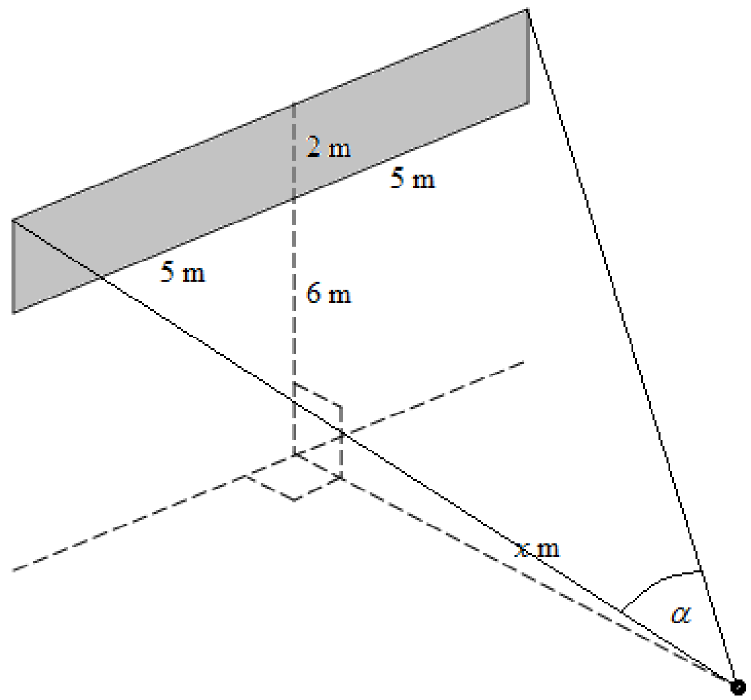
Part I
A large rectangular road sign (10 m by 2 m) is suspended 6 metres above a straight horizontal road.
A car travelling along the road approaches the road sign at right angle.
Consider the car as a point.
Let x m be the perpendicular distance of the car from the road sign, $x > 0$.
Refer to the following diagram for clarification.
● represents the car.



Question 1

a. α is the viewing angle of the upper width of the road sign as shown in the following diagram.

Show that $\alpha = \cos^{-1}\left(\frac{439}{489}\right)$ when $x = 20$. Do not use the result from Question 2a. 3 marks



b. Find the value of α, correct to 4 decimal places. 1 mark

Question 2 Refer to the diagram in Question 1.
The start of the road is 50 m from the road sign.

a. Show that $\alpha = \cos^{-1}\left(\dfrac{x^2+39}{x^2+89}\right)$ when the perpendicular distance of the car from the road sign is x.

3 marks

b. Sketch accurately the graph of α versus x. 3 marks

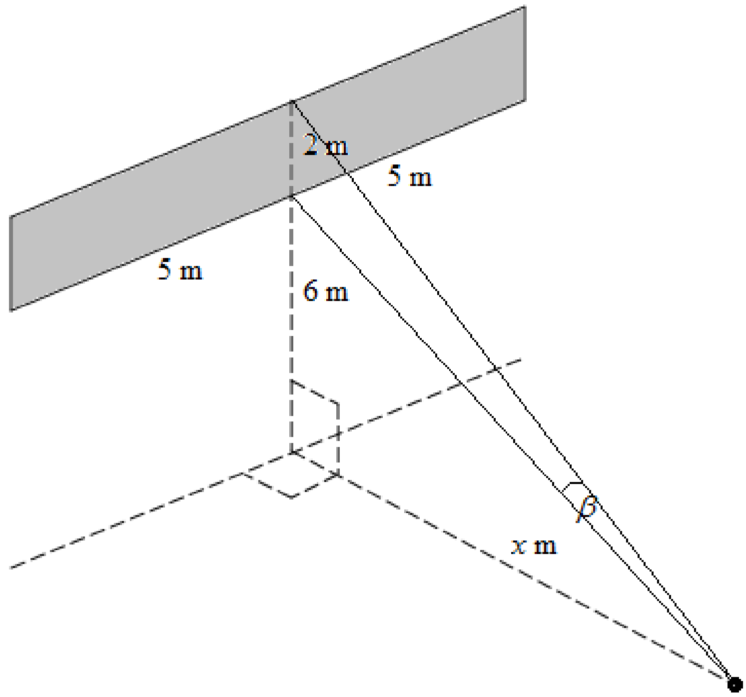
ci. State the exact domain of the graph of α versus x. 1 mark

cii. State the exact range of the graph of α versus x. 1 mark

Question 3

β is the viewing angle of the height of the road sign as shown in the following diagram.

Show that $\beta = \tan^{-1}\left(\frac{5}{56}\right)$ when $x = 20$. Do not use the result from Question 4a. 4 marks



Question 4 Refer to the diagram in Question 3.
The start of the road is 50 m horizontally from the road sign.

a. Show that $\beta = \tan^{-1}\left(\dfrac{2x}{x^2+48}\right)$ when the perpendicular distance of the car from the road sign is x.

3 marks

b. Sketch accurately the graph of β versus x.
Use CAS to find the coordinates of the turning point and end points, correct to 4 decimal places. 4 marks

ci. Find x in the form $m\sqrt{n}$ when β is maximum, where n is a prime number. Show differentiation.

3 marks

cii. Find the maximum value of β in simplest exact form. Show working. 2 marks

Question 5 The car approaches the road sign.

a. Show that the rate of change of viewing angle α with respect to distance x of the car from the road sign is given by $\frac{d\alpha}{dx} = \frac{-10x}{(x^2+89)\sqrt{x^2+64}}$. Do not use CAS. Show full working.

6 marks

b. The car travels at 20 m s^{-1}. Find the rate of change of viewing angle α with respect to time t seconds. Express your answer in terms of x.

2 marks

c. The car travels at 20 m s^{-1}. Find the rate of change of viewing angle α with respect to time t seconds when the car is 20 m horizontally from the road sign. Correct your answer to 4 decimal places.

1 mark

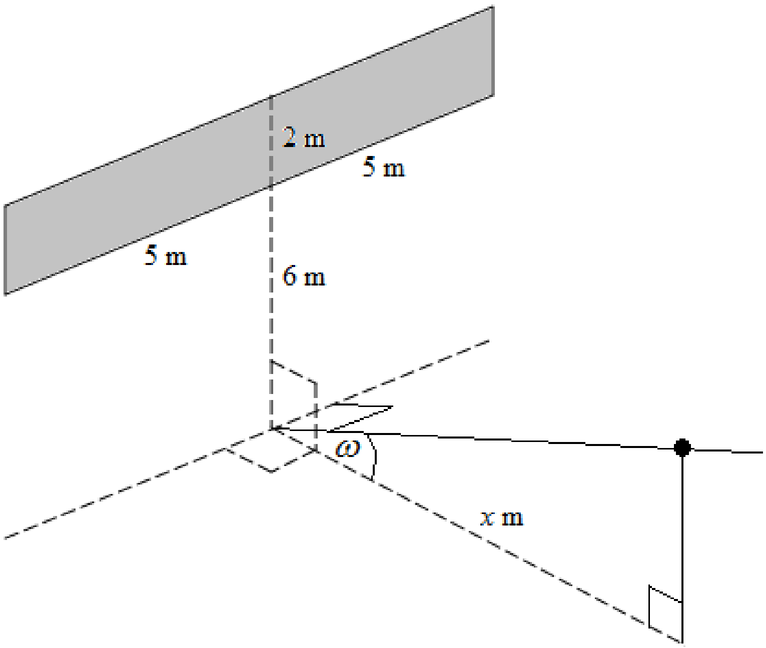
d. Assuming the car travels at constant speed 20 m s^{-1}, by CAS find the maximum rate of change of viewing angle α with respect to time t, and the distance x when it occurs. Correct your answers to 4 decimal places.

3 marks

Part II

Another rectangular road sign of the same dimension (10 m by 2 m) is suspended 6 metres above a straight road inclined at an angle of ω to the horizontal.

The car travelling along the road approaches the road sign at right angle.
Let x m be the ***horizontal*** perpendicular distance of the car from the road sign, $x > 0$.
Refer to the following diagram for clarification.
● represents the car.



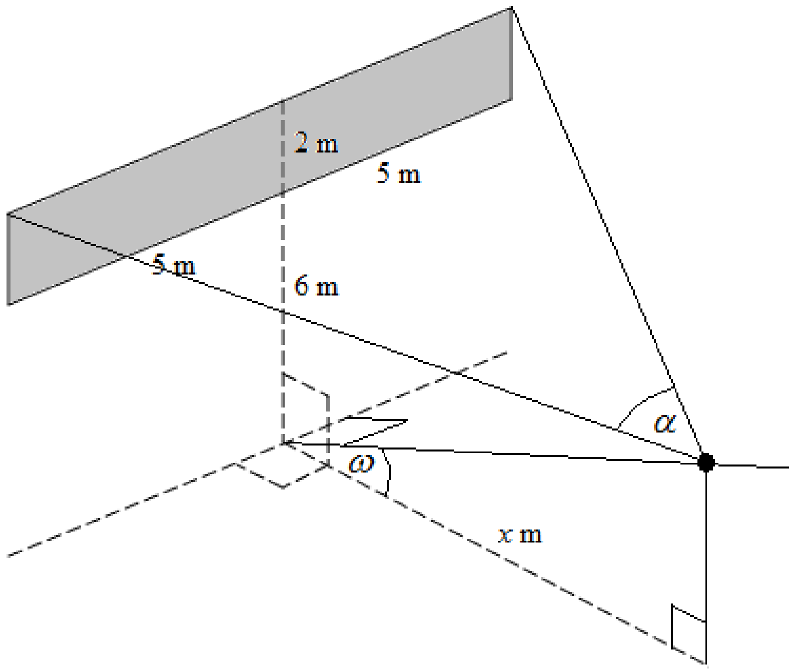
Question 6 Let $\omega = \frac{\pi}{8}$.

a. Show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$. 3 marks

α is the viewing angle of the upper width of the road sign as shown in the following diagram.

b. Find the value of α, correct to 4 decimal places, when $x = 20$. Do not use the result from Question 7a.

4 marks



Question 7 Refer to the diagram in Question 6.
The start of the road is 50 m horizontally from the road sign.

a. Show that $\alpha = \cos^{-1}\left(\dfrac{x^2 + \left(8 - \left(\sqrt{2} - 1\right)x\right)^2 - 25}{x^2 + \left(8 - \left(\sqrt{2} - 1\right)x\right)^2 + 25}\right)$ when the horizontal perpendicular distance of the car from the road sign is x. 4 marks

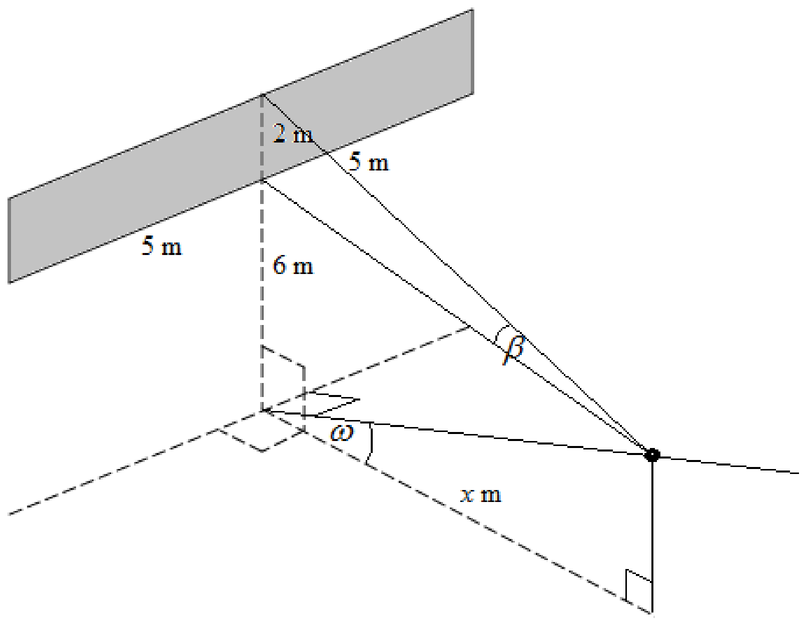
b. Sketch accurately the graph of α versus x.
Show coordinates of endpoints and turning points, correct to 4 decimal places. 4 marks

Question 8 Let $\omega = \dfrac{\pi}{8}$.

β is the viewing angle of the height of the road sign as shown in the following diagram.

a. Show that $\beta = \tan^{-1}\left(\dfrac{5\left(241+135\sqrt{2}\right)}{21631}\right)$ when $x = 20$. Do not use the result from Question 9a.

4 marks



b. Find the value of β, correct to 4 decimal places, when $x = 20$. 1 mark

Question 9 Refer to the diagram in Question 8. The start of the road is 50 m horizontally from the road sign.

a. Show that $\beta = \tan^{-1}\left(\frac{x}{\left(2-\sqrt{2}\right)x^2 - 7\left(\sqrt{2}-1\right)x + 24}\right)$ when the horizontal perpendicular distance of the car from the road sign is x. 4 marks

b. Sketch accurately the graph of β versus x.
Show coordinates of endpoints and turning points, correct to 4 decimal places. 4 marks

Question 10 Refer to the diagram in Question 8.
The start of the road is 50 m horizontally from the road sign. The car approaches the road sign along the road. For this question assume that the road inclines at a constant angle of ω to the horizontal.

a. Show that $\beta = \tan^{-1}\left(\frac{2x}{(1+\tan^2\omega)x^2 - (14\tan\omega)x + 48}\right)$ when the horizontal perpendicular distance of the car from the road sign is x.

3 marks

b. Sketch accurately the graphs of β versus x on the same set of axes for ***five*** different ***appropriate*** values of ω.

6 marks

c. Comment on the effects of changing ω on *the maximum value of* β, and the value of x when maximum β occurs. Support your comments with numerical values.

3 marks

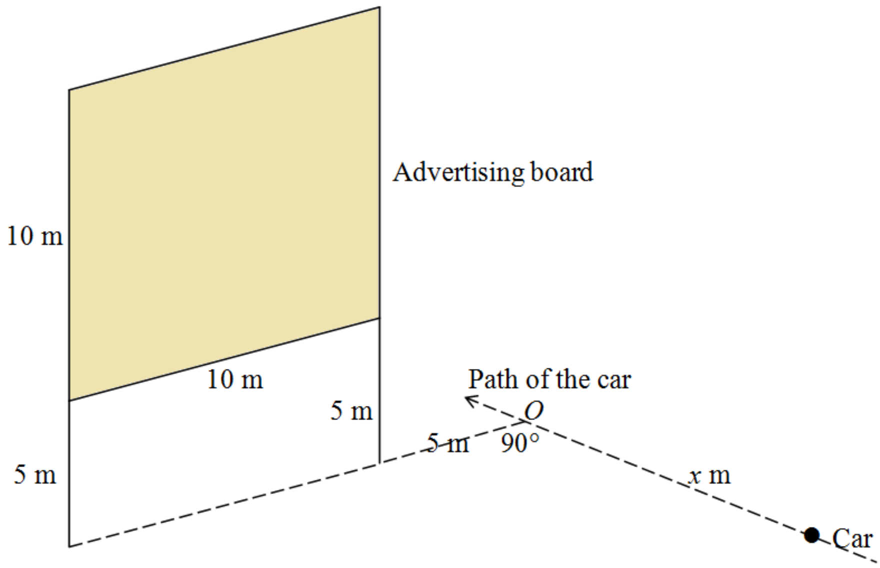
Part III

The car travels on another horizontal straight road. There is a large square (10 m by 10 m) advertising board erected upright with the lower edge of the board 5 m above the ground.

The closest horizontal distance of the board to the path of the car at O is 5 m.

The car is x metres from O, $x \geq 0$. See the diagram below for clarification.

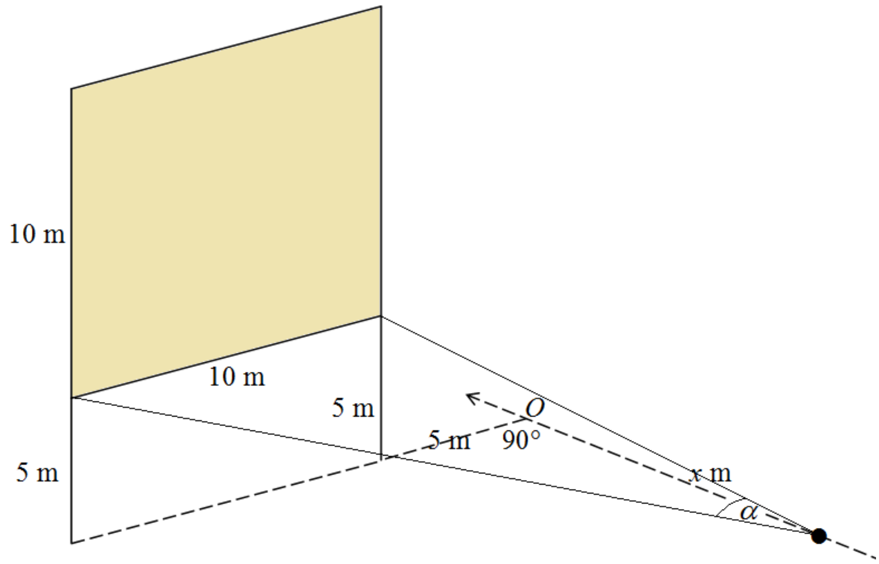
● represents the car.



Question 11

α is the viewing angle of the lower edge of the square board as shown in the following diagram.

Show that $\alpha = \cos^{-1}\left(\frac{10\sqrt{13}}{39}\right)$ when $x = 20$. Do not use the result from Question 12. 3 marks



Question 12 Refer to the diagram in Question 11. The start of the road is 50 m from O.

a. Show that $\alpha = \cos^{-1}\left(\frac{x^2 + 100}{\sqrt{(x^2 + 250)(x^2 + 50)}}\right)$ when the car is at distance x from O. 3 marks

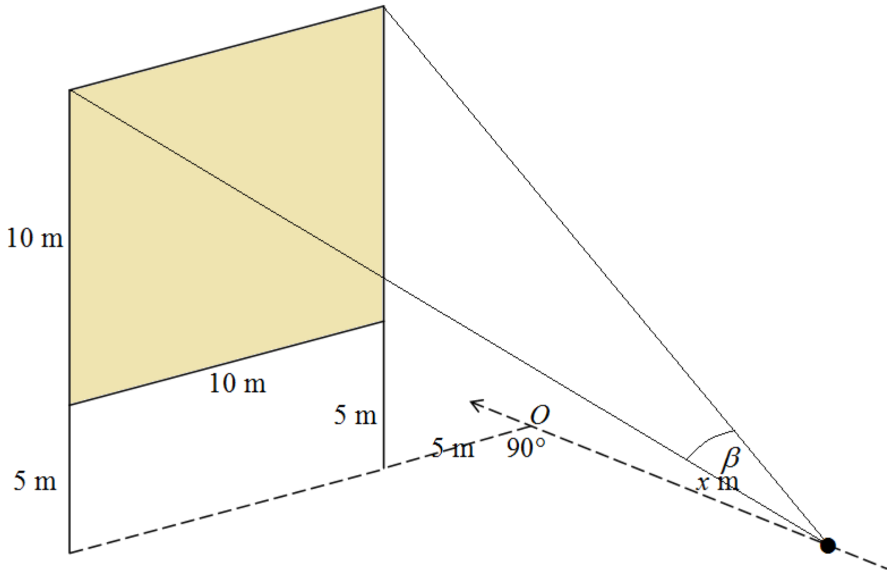
b. Sketch accurately the graph of α versus x.
Show coordinates of endpoints and stationary points, correct to 4 decimal places. 4 marks

ci. By CAS find the derivative of $\alpha = \cos^{-1}\left(\frac{x^2+100}{\sqrt{(x^2+250)(x^2+50)}}\right)$. 1 mark

cii Show that $x = 5\sqrt{2}$ when α has a maximum value. 2 marks

ciii. Find the maximum value of α in simplest exact form. 1 mark

Question 13 In the following diagram β is the viewing angle of the upper edge of the square board. The start of the road is 50 m from O.



a. Show that $\beta = \cos^{-1}\left(\dfrac{x^2 + 300}{\sqrt{(x^2 + 450)(x^2 + 250)}}\right)$ when the car is at distance x from O. 3 marks

b. Sketch accurately the graph of β versus x.
Show coordinates of endpoints and stationary points, correct to 4 decimal places. 4 marks

ci. Find the exact value of x when β is maximum. 1 mark

cii. Find the maximum value of β in simplest exact form. 1 mark

Question 14 Comment on the maximum values of α and β and the corresponding values of x. 2 marks

Question 15 Now consider the two vertical edges of the square board, one closer to the road (near edge) and one further from the road (far edge).
When the car is x m from O, the viewing angles of the vertical edges are θ and ϕ, and $\theta > \phi$.

a. Which vertical edge (near or far) has the viewing angle θ from the car? 1 mark

b. State the maximum values of θ and ϕ, and the corresponding values of x when they occur. 4 marks

c. Explain clearly how you arrived at your answers in part b. 2 marks

Question 16

a. Write down the rate of change of viewing angle α with respect to the distance x of the car from O.
Hint: Refer to Question 12 ci. 1 mark

b. The car travels at 20 m s^{-1}. Find the rate of change of the viewing angle α with respect to time t seconds. Express your answer in terms of x.

3 marks

c. The car travels at 20 m s^{-1}. Find the rate of change of the viewing angle α with respect to time t seconds when the car is 20 m from O. Express your answer in simplest exact form.

2 marks

d. The car travels at 20 m s^{-1}. What is the rate of change of the viewing angle α with respect to time t seconds when α is at its maximum value? Explain or show calculation.

2 marks

Part IV

A horizontal straight train track is perpendicular to a horizontal straight road.

The track crosses the road at crossing *O*.

The car travels along the road approaching crossing *O* at constant speed u m s^{-1}, $0 \le u \le 25$.

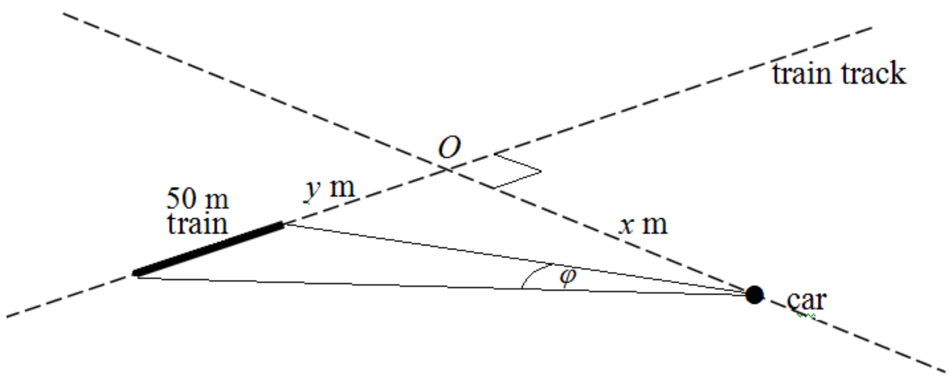
The train (50 m) moves along the track leaving crossing *O* at constant speed v m s^{-1}, $0 \le v \le \frac{5}{2}$.

At time $t = 0$, the car is 200 m from *O* and the **rear** of the train is at *O*.

At time $t > 0$, the car is x m from *O* and the **rear** of the train is y m from *O*.

At time $t > 0$, the viewing angle of the train from the car is φ .

Refer to the following diagram for clarification.



Question 17

Let $u = 25$ and $v = \frac{5}{2}$.

a. Find x (the distance of the car from O) and y (the distance of the rear of the train from O) at $t = 3$. 2 marks

b. Find the simplest exact value of φ at $t = 3$. Do not use the result from Question 18. 3 marks

c. Is φ increasing or decreasing at $t = 3$?
Explain your answer by calculating φ at two other appropriate times. 3 marks

Question 18

a. At time $t > 0$, the car is x m from O and the rear of the train is y m from O.

Show that $\varphi = \tan^{-1}\left(\dfrac{50x}{x^2 + y^2 + 50y}\right)$. 3 marks

b. Find x and y in terms of u, v and/or t. Hence show that $\varphi = \tan^{-1}\left(\dfrac{50(200 - ut)}{(200 - ut)^2 + v^2t^2 + 50vt}\right)$. 2 marks

Let $u = 10v$.

c. Show that $\varphi = \tan^{-1}\left(\dfrac{500(20 - vt)}{40000 - 3950vt + 101v^2t^2}\right)$. 1 mark

d. On the same set of axes, sketch ***five*** φ versus t graphs using five appropriate values of v, $0 \le v \le \frac{5}{2}$.
Coordinates, correct to 4 decimal places, of endpoints and turning points are required. 8 marks

e. Comment on the effects of changing the values of v on the graphs in relation to endpoints and turning points.

4 marks

Question 19

Consider $u = 10v$.

a. By CAS or other means find $\frac{d\varphi}{dt}$ in terms of t and v. 2 marks

b. Show that the maximum value of φ occurs when $vt = 20 - 10\sqrt{\frac{14}{101}}$. 3 marks

c. Hence find the maximum value of φ, correct to 4 decimal places. 2 marks

The speed of the car is u, and the speed of the train is v. Let $u = 25$ and $v = \frac{5}{2}$.

At time $t = 0$, the car is 200 m from *O* and the **rear** of the train is at *O*.
At time $t > 0$, the car is *x* m from *O* and the **rear** of the train is *y* m from *O*.
At time $t > 0$, the viewing angle of the train from the car is φ.
Let t_O be the time when the car passes crossing *O*.
The car passes crossing *O* and continues on its trip.

Question 20

a. Find t_O. 1 mark

b. Find y when the car passes crossing *O*. 1 mark

c. Show that $\varphi = \tan^{-1}\left(\dfrac{200(t-8)}{100(t-8)^2 + t + 20}\right)$ when $t \geq t_O$. 4 marks

d. Find t (correct to 4 decimal places) when the value of φ is maximum. 1 mark

End of task

