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2017 Specialist Mathematics

Year 12 Problem Solving Task

Time allowed: 2 hours plus

You are allowed: 1 bounded reference, 1 CAS, 1 scientific calculator
Working must be shown for questions worth 2 or more marks. Total: 80 marks

Problem Solving Task

Theme: Vertical motion

Time is measured in seconds, length is in metres, mass is in kilogram and force is in newtons.
Include units in your answers.
Correct your answers to 2 decimal places unless stated otherwise.

Part I

Question 1 *Upward and downward motions of a particle over a short distance*

A 0.010-kg particle is projected vertically upwards with a speed of 20 m s^{-1} from the ground level at $t = 0$.
Assume that the gravitational field is $g = 9.83$ N kg^{-1} throughout its motion, and air resistance is negligible.

a. Find the weight of the particle. 1 mark

b. What is the acceleration of the particle? 1 mark

c. Determine the maximum height reached by the particle. 2 marks

d. Determine the time taken by the particle to reach the maximum height. 1 mark

e. What is the time taken by the particle to fall back to the ground? 1 mark

f. Calculate the speed of the particle just before it hits the ground. 2 marks

Question 2 *Upward and downward motions of a particle over a short distance with air resistance*

A 0.010-kg particle is projected vertically upwards with a speed of 20 m s^{-1} from the ground level at $t = 0$.
Assume that the gravitational field is $g = 9.83$ N kg^{-1} throughout its motion.

The force of air resistance on the particle is given by $0.0005v^2$ newtons where v m s^{-1} is the speed of the particle at $t > 0$.

a. Draw a labeled force diagram to show the forces on the particle (•) on its upward motion. 1 mark

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b. Let x be the distance from the ground at $t > 0$.

Show that the equation of motion of the particle is $\frac{d^2x}{dt^2} = -\frac{v^2 + 196.6}{20}$. 2 marks

c. Show that $x = 10\log_e\left(\frac{596.6}{v^2 + 196.6}\right)$. 4 marks

d. Show that the maximum height reached by the particle is 11.100757 m 1 mark

e. Show that $v = \sqrt{596.6e^{-\frac{x}{10}} - 196.6}$ while the particle is moving upwards. 2 marks

f. Show that the time taken to reach the maximum height is 1.3679 s. 2 marks
Hint: Use CAS

Now consider the return journey, i.e. the downward motion of the particle.
Let x be the distance from the highest point and v the speed of the particle at $t > 0$.
At $t = 0$, $x = 0$ and $v = 0$.
The force of air resistance on the particle is the same as that for the upward motion, i.e. $0.0005v^2$.

g. Show that the equation of motion of the particle is $\frac{d^2x}{dt^2} = \frac{196.6 - v^2}{20}$. 2 marks

h. Show that $v = 14.0214\sqrt{1 - e^{-\frac{x}{10}}}$ while the particle is moving downwards. 2 marks

i. Calculate the speed of the particle just before it hits the ground. 1 mark

j. Determine the total time for the upward and downward motion of the particle. 3 marks

Question 3 *Upward motion of a particle under the influence of gravity only*

A 0.010-kg particle is projected vertically into space from the surface of the earth at $t = 0$.
At $t \geq 0$ it moves under the influence of gravity only.
Take the distance from the centre to the surface of the earth as 6.37×10^6 m.
The force of gravity on the particle is given by $\frac{3.99 \times 10^{12}}{x^2}$ newtons for $x \geq 6.37 \times 10^6$ where x is the distance of the particle from the centre of the earth.

a. Show that the equation of motion is $\frac{d^2x}{dt^2} = -\frac{3.99 \times 10^{14}}{x^2}$ for $t \geq 0$. 1 mark

b. Use $\frac{d^2x}{dt^2} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$ to show that $v^2 - {v_0}^2 = \frac{7.98 \times 10^{14}}{x} - 1.253 \times 10^8$, given $v = v_0$ at $t = 0$. 4 marks

c. Show that the speed of projection v_0 of the particle for it to escape from the earth is 1.119×10^4 m s^{-1}.
Hint: Let $x \to \infty$ and $v \to 0$.

1 mark

d. Show that $x = a(t+b)^{\frac{2}{3}}$ is a solution to $\frac{d^2x}{dt^2} = -\frac{3.99\times10^{14}}{x^2}$, where a and b are real constants. 2 marks

e. If $v_0 = 1.119\times10^4$ at $t = 0$, find the values of constants a and b. 2 marks

f. Calculate the maximum distance of the particle from the centre of the earth if $v_0 = 2.00\times10^3$ m s^{-1}.

2 marks

Time is measured in seconds, length is in metres, mass is in kilogram and force is in newtons.
Include units in your answers.
Correct your answers to 2 decimal places unless stated otherwise.
Assume that the gravitational field is $g = 9.83$ N kg^{-1} throughout the motion.

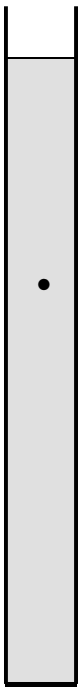
Part II

Question 4 *Downward motion of a particle through a viscous fluid over a short distance*

A 0.010-kg particle is released from rest in a viscous fluid at $t = 0$.
Assume that the gravitational field is $g = 9.83$ N kg^{-1} throughout its motion.
The resistive force of the fluid on the particle is given by $0.01966v$ newtons where v m s^{-1} is the speed of the particle at $t > 0$. No other forces act on the particle.

a. Draw a labeled force diagram to show the forces on the particle (•) while it is falling through the fluid.

1 mark



b. Let x be the distance at $t > 0$ from the releasing point.

Show that the equation of motion of the particle is $\frac{d^2x}{dt^2} = 1.966(5 - v)$. 2 marks

c. Find the maximum speed of the particle in the fluid. 1 mark

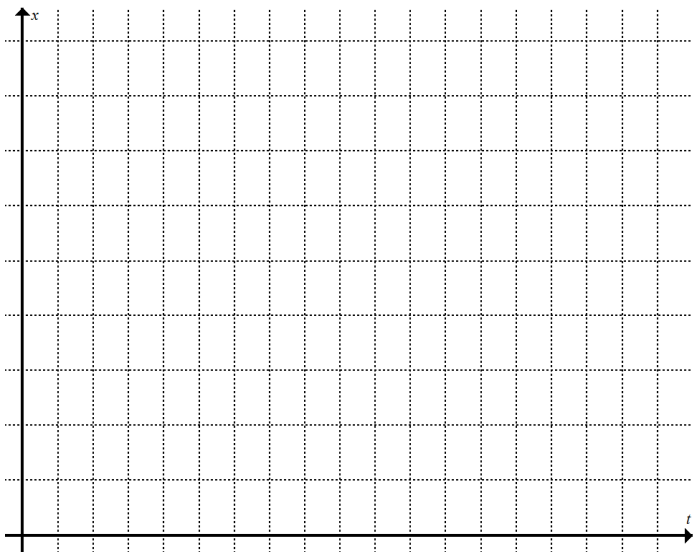
d. Show that $v = 5\left(1 - e^{-1.966t}\right)$ at $t > 0$. 2 marks

e. Find the exact value of $e^{-1.966t}$ and hence the value of t when the particle reaches 90% of its maximum speed.

2 marks

f. Find x in terms of t. 2 marks

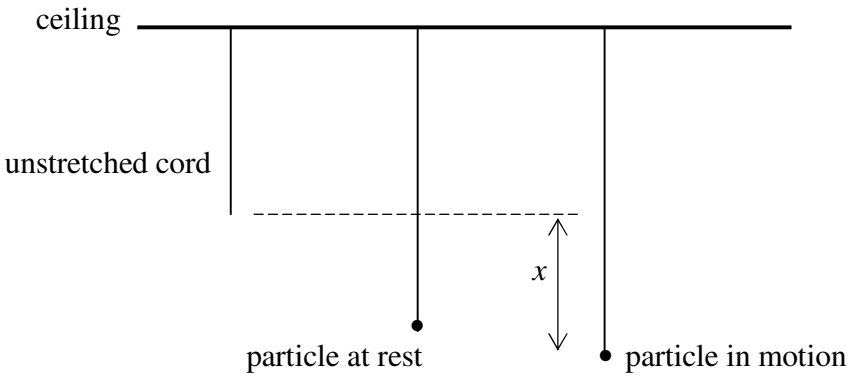
g. Draw the graph of x versus t for $0 \le t \le 1.5$. Show coordinates of the right endpoint. 2 marks



Question 5 *Motion of a particle suspended by an elastic cord of negligible mass*

A 0.010-kg particle is attached to a vertical elastic cord fastened to the ceiling.
When the particle is at rest, the cord is stretched by the weight of the particle and extended by 0.20 m.
When the cord is stretched by x m, the force on the particle is $-0.4915x$ newtons.
The particle is pulled down further by 0.10 m and released.
Assume that the gravitational field is $g = 9.83$ N kg^{-1} throughout its motion, and air resistance is negligible.

a. Show and label the forces on the particle while it is in motion. 1 mark



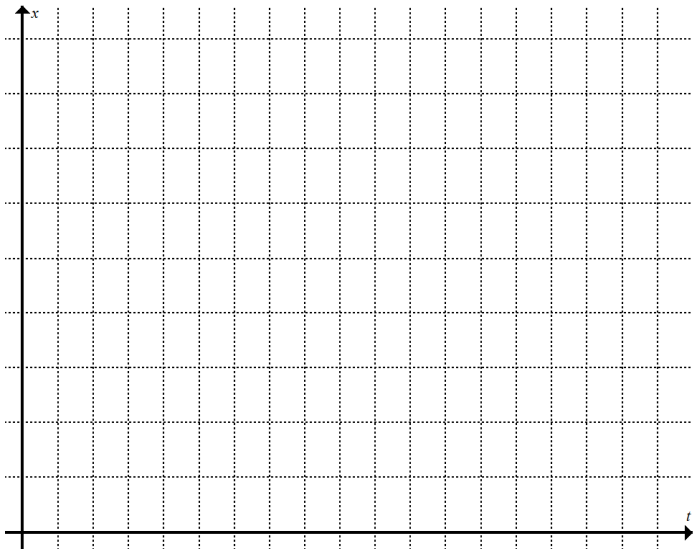
b. Show that $\frac{d^2x}{dt^2} = -49.15(x - 0.2)$. 2 marks

c. Show that $v^2 = 49.15(x - 0.1)(0.3 - x)$. 2 marks

d. Use the result in part c to find the possible values of x. 2 marks

e. Verify that $x = 0.10\cos\left(\sqrt{49.15}\, t\right) + 0.20$ is a solution to the differential equation in part b. 2 marks

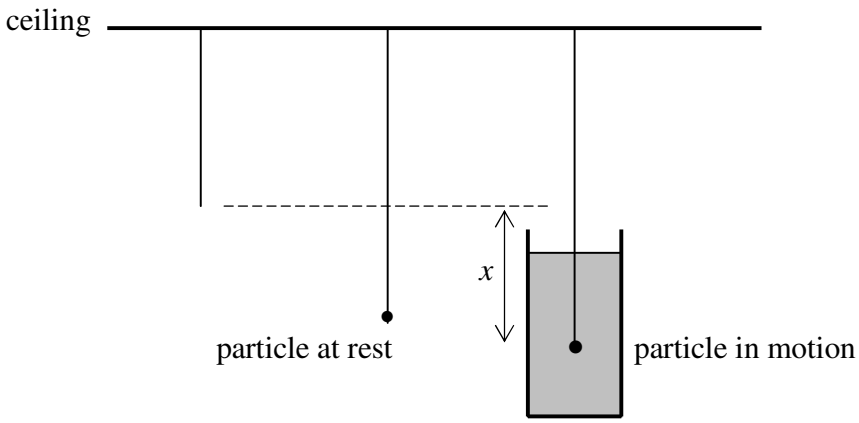
f. Draw a graph of x versus t for $0 \le t \le \dfrac{3\pi}{\sqrt{49.15}}$. 2 marks



Question 6 *Motion in a viscous fluid of a particle suspended by an elastic cord of negligible mass*

A 0.010-kg particle is attached to a vertical elastic cord fastened to the ceiling.
When the particle is at rest, the cord is stretched by the weight of the particle and extended by 0.20 m.
When the cord is stretched by x m, the force on the particle is $-0.4915x$ newtons.
The particle is pulled down further by 0.10 m and released.
Assume that the gravitational field is $g = 9.83$ N kg^{-1} throughout its motion.
The resistive force of the fluid on the particle is given by $0.01966v$ newtons where v m s^{-1} is the speed of the particle at $t > 0$. **No other forces act on the particle**.

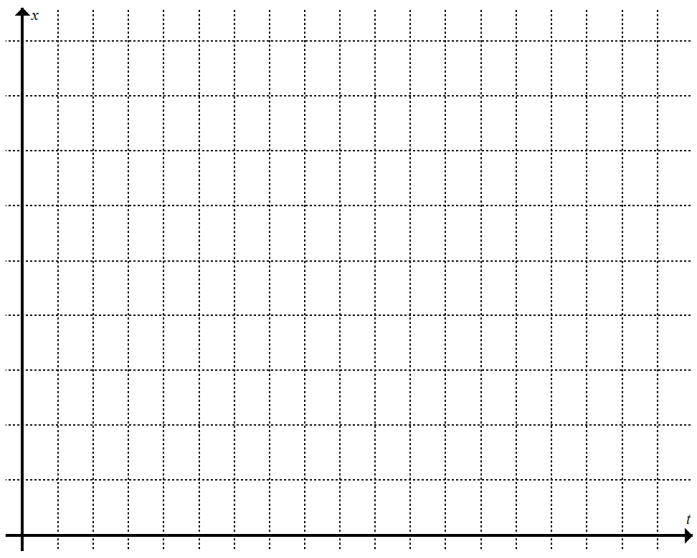
a. Show and label the forces on the particle while it is in motion. 2 marks



b. Show that $\frac{d^2x}{dt^2} + 1.966\frac{dx}{dt} + 49.15(x - 0.2) = 0$ 3 marks

c. Verify that $x = 0.101e^{-0.983t}\cos(6.941t - 0.141) + 0.20$ is a solution to the differential equation $\frac{d^2x}{dt^2} + 1.966\frac{dx}{dt} + 49.15(x - 0.2) = 0$. 6 marks

d. Draw the graphs of $x = 0.101e^{-0.983t}\cos(6.941t - 0.141) + 0.20$, $x = -0.101e^{-0.983t} + 0.20$ and $x = 0.101e^{-0.983t} + 0.20$ on the same set of axes for $0 \le t \le 3$. 4 marks



e. Comparing the oscillations in this question with the oscillations in Question 5, which one takes longer to complete one oscillation. Show calculations to support your answer.

2 marks

End of task