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2017 VCAA Specialist Mathematics Exam 1 Solutions

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Q1a $3xy^2+2y=x$, $\frac{d}{dx}\left(3xy^2+2y\right)=1$, $6xy\frac{dy}{dx}+3y^2+2\frac{dy}{dx}=1$, $(6xy+2)\frac{dy}{dx}+3y^2=1$. At $(1,-1)$, $\frac{dy}{dx}=\frac{1}{2}$

.: Tangent: $y+1=\frac{1}{2}(x-1)$, $x-2y=3$

Q2 $\int_1^{\sqrt{3}}\frac{1}{x\left(1+x^2\right)}dx=\int_1^{\sqrt{3}}\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx=\int_1^{\sqrt{3}}\frac{1}{x}dx-\int_1^{\sqrt{3}}\frac{x}{1+x^2}dx$

$=\int_1^{\sqrt{3}}\frac{1}{x}dx-\int_2^4\frac{1}{2}.\frac{1}{u}du$ where $u=1+x^2$

$=\left[\log_e x\right]_1^{\sqrt{3}}-\frac{1}{2}\left[\log_e u\right]_2^4=\log_e\sqrt{3}-\log_e\sqrt{2}=\log_e\left(\sqrt{\frac{3}{2}}\right)$

Q3 Conjugate root: Given $1-i$ is a root (solution), .: $1+i$ is also a root. Let b be the third root.
Sum of roots $=$ product of roots $=-a$
.: $(1-i)+(1+i)+b=(1-i)(1+i)b$
.: $2+b=2b$, .: $b=2$
The other 2 solutions are 2 and $1+i$.

Alternatively, let $(z-1+i)(z-1-i)(z-b)=z^3+az^2+6z+a$
Compare the coefficient of z^2 term and the constant term to find $b=2$.

Q4 Normal distribution of the population: $\mu=298$ and $\sigma=3$
.: normal distribution of $\overline{X}$:
$\mathrm{E}\left(\overline{X}\right)=\mu=298$ and $\mathrm{sd}\left(\overline{X}\right)=\frac{\sigma}{\sqrt{n}}=\frac{3}{2}=1.5$
$\Pr\left(\overline{X}<295\right)=\Pr\left(\overline{X}<(298-2\times1.5)\right)\approx0.025$

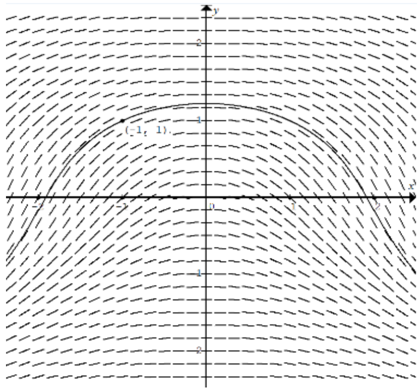
Q5 $\overrightarrow{CB}=\tilde{b}-\tilde{c}=-\tilde{i}+\tilde{k}$, $\overrightarrow{CD}=\tilde{d}-\tilde{c}=(a-2)\tilde{i}-\tilde{j}-\tilde{k}$
$\overrightarrow{CB}.\overrightarrow{CD}=\left|\overrightarrow{CB}\right|\left|\overrightarrow{CD}\right|\cos\frac{\pi}{3}$, .: $-(a-2)-1=\sqrt{2}\sqrt{(a-2)^2+1+1}\times\frac{1}{2}$
.: $1-a=\sqrt{\frac{a^2-4a+6}{2}}$, .: $1-a\ge0$ and $a^2-4a+6\ge0$, .: $a\le1$
Solve $1-a=\sqrt{\frac{a^2-4a+6}{2}}$ by squaring both sides, .: $a^2=4$.
Since $a\le1$, .: $a=-2$

Q6 $f(x)=\left(\sin^{-1}x\right)^{-1}$, $f'(x)=-\frac{1}{\sqrt{1-x^2}\ \sin^{-1}x}$ by the chain rule
.: $1-x^2>0$ and $\sin^{-1}x\ne0$
.: $-1<x<1$ and $x\ne0$, .: the largest set of values of x for which $f'(x)$ is defined is $\{x:-1<x<1,\ x\in R\}\setminus\{0\}$.

Q7 $x=\cos^3 t$, $x'=-3\cos^2 t\sin t$: $y=\sin^3 t$, $y=3\sin^2 t\cos t$
.: $(x')^2+(y')^2=9\sin^2 t\cos^2 t\left(\cos^2 t+\sin^2 t\right)=9\sin^2 t\cos^2 t$
.: $\sqrt{(x')^2+(y')^2}=3\sin t\cos t=\frac{3}{2}\sin 2t$

Length $=\int_0^{\frac{\pi}{4}}\sqrt{(x')^2+(y')^2}\,dt=\int_0^{\frac{\pi}{4}}\frac{3}{2}\sin 2t\,dt=\left[-\frac{3}{4}\cos 2t\right]_0^{\frac{\pi}{4}}=\frac{3}{4}$

Q8a $x\approx1.9$



Q8b $\frac{dy}{dx}=\frac{-x}{1+y^2}$, $\int\left(1+y^2\right)dy=\int -x\,dx$, $y+\frac{y^3}{3}=-\frac{x^2}{2}+c$

Use $(-1,1)$ to find $c=\frac{11}{6}$, .: $y+\frac{y^3}{3}=-\frac{x^2}{2}+\frac{11}{6}$

.: $2y^3+6y+3x^2-11=0$

Q9a $\vec{F}=m\tilde{a}=2\times\frac{43\tilde{i}-18\tilde{j}-\left(3\tilde{i}+2\tilde{j}\right)}{10}=2\left(4\tilde{i}-2\tilde{j}\right)=8\tilde{i}-4\tilde{j}$

$\left|\vec{F}\right|=\sqrt{8^2+(-4)^2}=4\sqrt{5}$ newtons

Q9b $\tilde{a}=4\tilde{i}-2\tilde{j}$, $\tilde{v}=\int_0^t\left(4\tilde{i}-2\tilde{j}\right)dt+3\tilde{i}+2\tilde{j}=(4t+3)\tilde{i}-(2t-2)\tilde{j}$

$\tilde{s}=\int_0^{10}\left((4t+3)\tilde{i}-2(t-1)\tilde{j}\right)dt=\left[\frac{(4t+3)^2}{8}\tilde{i}-(t-1)^2\tilde{j}\right]_0^{10}=230\tilde{i}-80\tilde{j}$

Q10a $\frac{d}{dx}\left(x\cos^{-1}\left(\frac{x}{a}\right)\right)=\cos^{-1}\left(\frac{x}{a}\right)-\frac{x}{\sqrt{a^2-x^2}}$ by the product rule

Q10b $\cos^{-1}\left(\frac{x}{2}\right)\ge0$, $-2\le x\le2$, maximal domain is $[-2,2]$, and the range is $\left[0,\sqrt{\pi}\right]$

Q10c $V=\int_{-2}^{2}\pi y^2\,dx=\pi\int_{-2}^{2}\cos^{-1}\left(\frac{x}{2}\right)dx$

$=\pi\int_{-2}^{2}\left(\frac{d}{dx}x\cos^{-1}\left(\frac{x}{2}\right)+\frac{x}{\sqrt{2^2-x^2}}\right)dx$ by part a

$=\pi\int_{-2}^{2}\frac{d}{dx}x\cos^{-1}\left(\frac{x}{2}\right)dx+\pi\int_{-2}^{2}\frac{x}{\sqrt{2^2-x^2}}dx=\pi\left[x\cos^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}=2\pi^2$

Note: $\int_{-2}^{2}\frac{x}{\sqrt{2^2-x^2}}dx=0$ because $\frac{x}{\sqrt{2^2-x^2}}$ is an odd function.

Please inform mathline@itute.com re conceptual and/or mathematical errors.