# 2018 Further Mathematics

# Year 12 Modelling Task

**Time allowed: 2 hours plus**

# Theme: Population growth/decline

**Rationale:** A government needs to know the growth/decline of the population for infrastructure planning.

**Required skills/knowledge:** Setting up matrices, matrix operations and the use of first-order linear matrix recurrence relations to model a situation.

**The task:** From the given information about the changes in a population, a matrix is developed to predict the yearly change in the size of the population in different parts of a country over a period.

The effects of government policies on the population are investigated. Students alter the parameters to explore the consequences

**Part I:**

People in a certain country settle in three areas: Capital cities, A, regional cities, B and country towns, C. The populations in the three areas (A, B and C) can change due to migration of people from one area to another, natural causes (birth and death), and immigrants from other countries.

At the start of year 2019, the country will have 25 million people. One tenth is in country towns and 30% are in regional cities. The rest are in capital cities.

(1) *Show that the population (in millions) distribution matrix at the start of year 2019 is*

$$\begin{matrix} A \\ B \\ C \end{matrix}\begin{bmatrix} 15 \\ 7.5 \\ 2.5 \end{bmatrix}$$

In year 2019, 4% of capital city population migrates to regional cities and 0.1% to country towns.
5% of regional city population migrates to capital cities and 0.2% to country towns.
8% of country town population migrates to regional cities and 2% to capital cities.
The birth rates and death rates of A, B and C are show in the following table.

| | A | B | C |
|---|---|---|---|
| Birth rate (% of each population) | 1.0 | 1.2 | 1.5 |
| Death rate (% of each population) | 1.5 | 1.2 | 1.0 |

For example, in the shaded cell 1.0 means 1.0% of the total capital city population.

(2) *Show that the total **capital city** population at the end of year 2019 is 14.735 million.*

(3) *By quoting some percentages and without calculations indicate and explain whether the* ***country's*** *population is increasing or decreasing.*

(4) *By quoting some numbers and percentages, and without calculations, indicate whether the country's population annual change is accelerating or decelerating.*

The given information is reprinted below for your convenience.

At the start of year 2019, the country will have 25 million people. One tenth is in country towns and 30% are in regional cities. The rest are in capital cities.
In year 2019, 4% of capital city population migrates to regional cities and 0.1% to country towns.
5% of regional city population migrates to capital cities and 0.2% to country towns.
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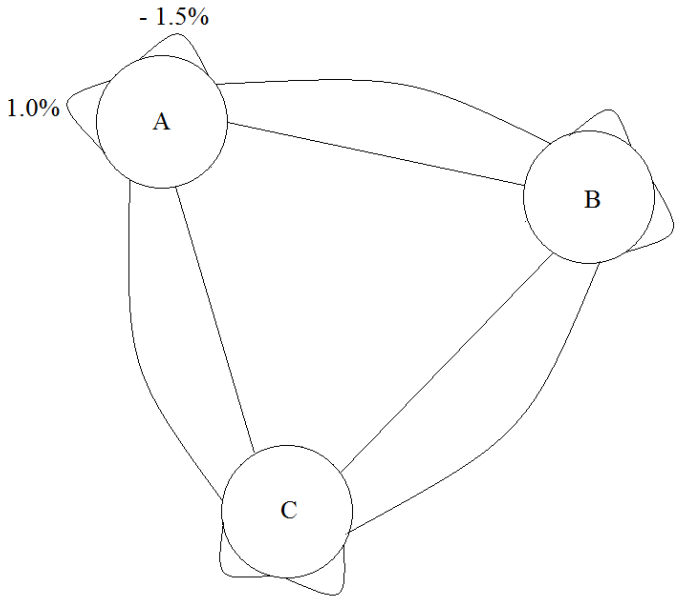
(5) *Complete the following diagram by drawing arrowhead on each line for migrations and writing a percentage next to each arrowhead to indicate the changes in year 2019. Use a negative sign for death.*



(6) *Use the diagram to construct a matrix summarising the percentage changes (expressed as decimals) in populations of A, B and C in year 2019. Two entries are shown.*

| | A | B | C | |
|---|---|---|---|---|
| ⎡ | 0.954 | _____ | _____ | ⎤ A |
| ⎢ | _____ | _____ | 0.080 | ⎥ B |
| ⎣ | _____ | _____ | _____ | ⎦ C |

(7) *Find the population in each of A, B and C at the start of 2020.*

(8) *Find the population in each of A, B and C at the start of 2021, assuming the percentages remain the same in year 2020.*

(9) *Are your answers in (3) and (4) justified by the results in (7) and (8)?*

The government wishes to at least maintain the country's population at 25 million people.
Some suggestions were made by the ministers.

**Suggestion W**: Recruit 100,000 migrants from other countries at the end of each year and settle them in the capital cities.

(10) *Find the population in each of A, B and C at the start of 2020.*

(11) *Find the population in each of A, B and C at the start of 2021, assuming the percentages remain the same in year 2020.*

Let $S_0 = \begin{bmatrix} 15 \\ 7.5 \\ 2.5 \end{bmatrix}$ be the population distribution matrix for the start of year 2019, $S_1$ for the start of year 2020, $S_2$ for the start of year 2021 etc.

(12) *Write down a complete matrix formula connecting $S_n$ and $S_{n+1}$ for suggestion W.*

**Suggestion X**: Recruit 100,000 migrants from other countries at the end of each year.
50,000 are to be settled in the regional cities, and 50,000 are to be settled in the country towns.

(13) *Find the population in each of A, B and C at the start of 2020 and 2021, assuming the percentages remain the same in year 2020.*

(14) *Discuss which suggestion, W or X, best meet the government's objective in the short term.*

(15) *Discuss the long term effects on population distribution for each suggestion.*

(16) *Suggest a way to settle the 100,000 immigrants in order to maintain the same proportion of capital city, regional city and country town populations. Rate the goodness of your suggestion.*

**Part II: Investigation**

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| Death rate (% of each population) | 1.5 | 1.2 | 1.0 |

The government wishes to at least maintain the country's population at 25 million people.
Some suggestions were made by the ministers.

**Suggestion W**: Recruit 100,000 migrants from other countries at the end of each year and settle them in the capital cities.

**Suggestion X**: Recruit 100,000 migrants from other countries at the end of each year.
50,000 are to be settled in the regional cities, and 50,000 are to be settled in the country towns.

**Suggestion Y**: Encourage bigger family to increase birth rate by a factor of $\alpha$ in the capital cities.
For example, if the birth rate is 1.0% and $\alpha = 1.01$, the new birth rate is $1.01 \times 1.0\% = 1.01\%$.

(1) *Choose at least 8 suitable/reasonable/appropriate values of $\alpha$ for your investigation of the effects on the population distribution in the country over 5 years (start of year 2019 to start of year 2024).*

*Summarise your results in the following table.*

| *Start of* / $\alpha$ | *2020* | *2021* | *2022* | *2023* | *2024* |
|---|---|---|---|---|---|
| | A [ ____ ]<br>B [ ____ ]<br>C [ ____ ] | | | | |
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(2) *Write comments related to population decline and growth, explosion, fluctuations, validity of your chosen values of* $\alpha$*, the best value of* $\alpha$ *in meeting the government's objective.*