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2018 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

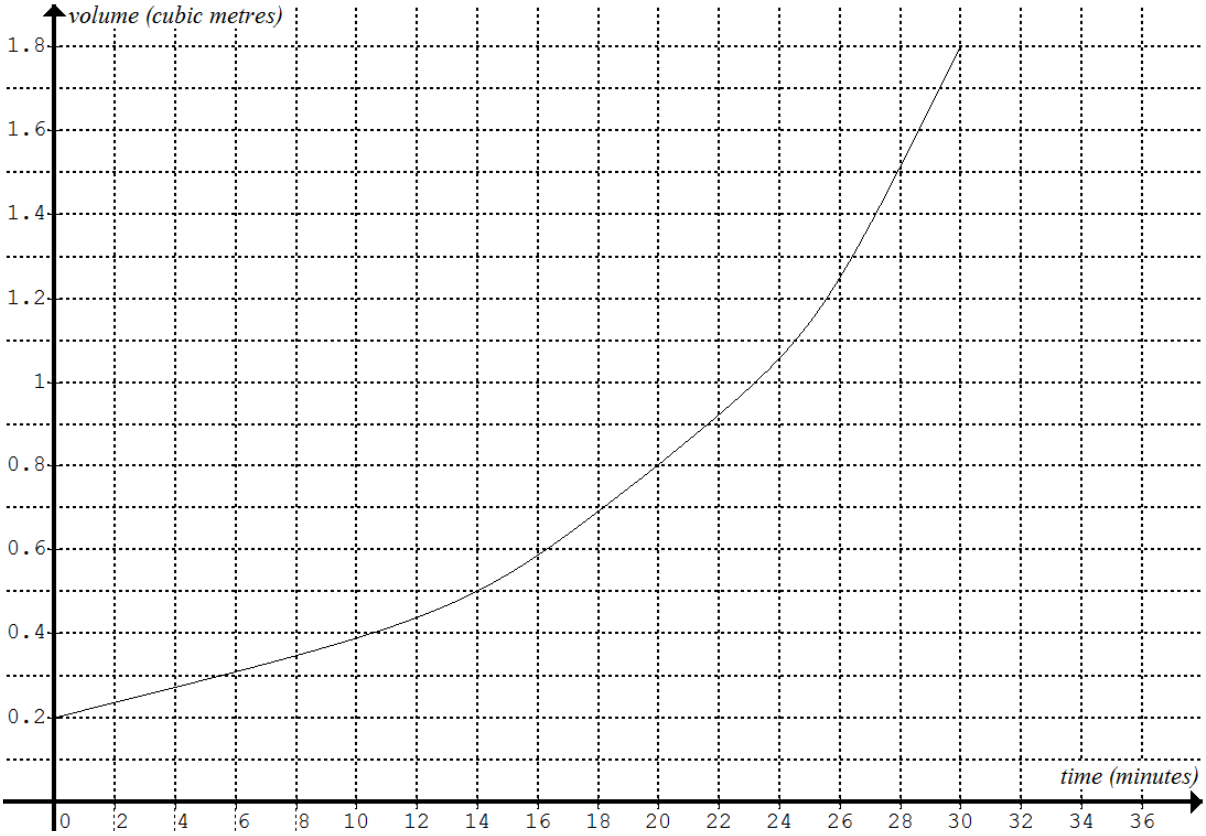
Modelling Task

Time allowed: 2.5 hours plus

Application Task: Parts I, II and III
Modelling Task: Parts I and II only OR Parts I and III only

Theme: Curve fitting

Rain falling on the roof of a house during a storm was collected in a storage tank for watering garden and general cleaning.
Over a 30-minute period ($t = 0$ to $t = 30$) the volume V (m^3) of rainwater in the tank was measured every minute.
The data points were plotted and a smooth curve was drawn.
The following graph shows only the smooth curve drawn without the data points.



Your task is to use different mathematical functions to model the curve.

Part I (50 marks)

a Use the following piecewise function to model the curve. Determine the values of the coefficients a, b, c, d, e and f.

6 marks

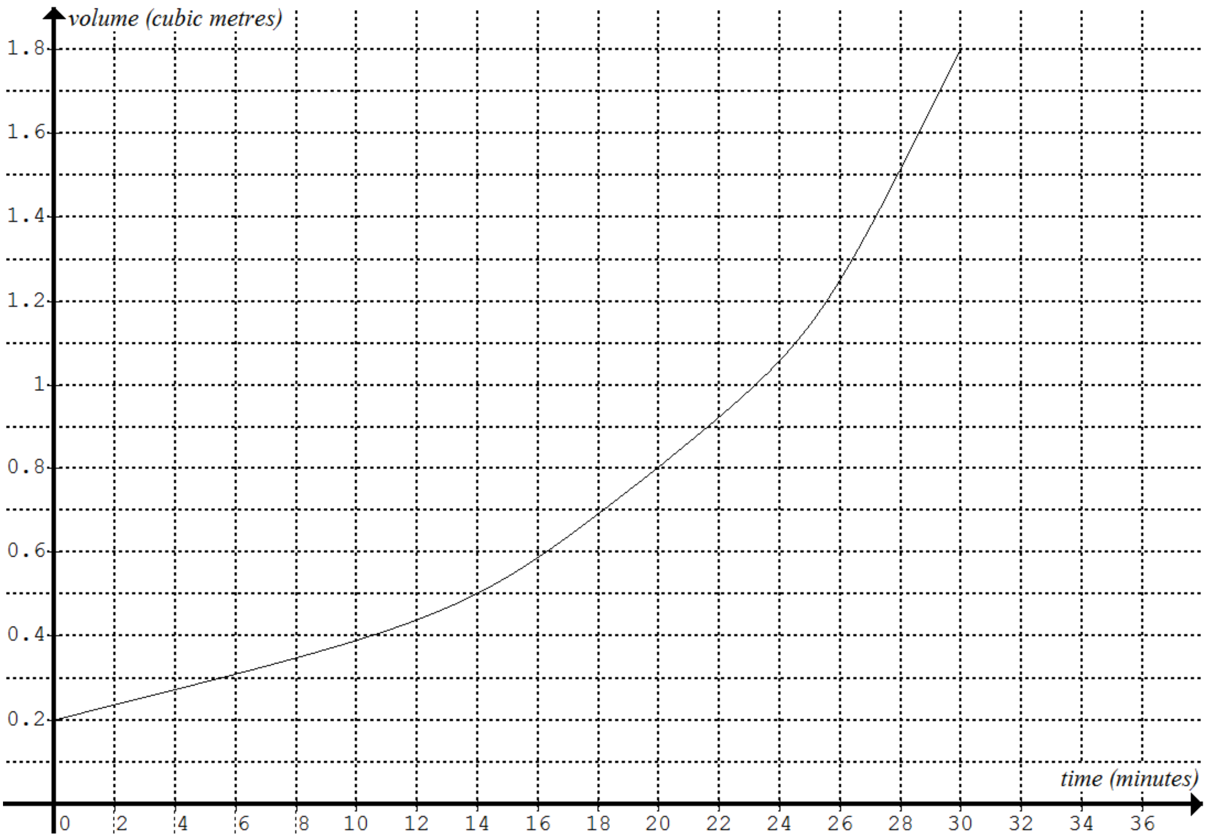
$$V = \begin{cases} at + b & 0 \le t < 14 \\ ct + d & 14 \le t < 24 \\ et + f & 24 \le t \le 30 \end{cases}$$

b. Sketch the piecewise function on the same set of axes of the volume-time graph on page 2. 3 marks

c. Determine the average rate of increase in the volume for each interval. 3 marks

d. Use 5 appropriate straight-line pieces to model the original curve. Choose an appropriate interval for each piece to best model the curve and determine the piecewise function similar to that in part I a. Sketch your piecewise function in the following diagram.

12 marks



e. Which one of the two piecewise straight-line functions gives the best fit to the original curve? How did you decide which one is the best fit?

2 marks

f. Discuss briefly one disadvantage related to the rate of change, in using piecewise straight-line functions to model the original curve.

2 marks

g. To avoid the problem discussed in Part I f, a power function of the type shown below, before and after appropriate transformations, is used to model the original curve.

Before transformations: $y = x^n$, $n \in Q$

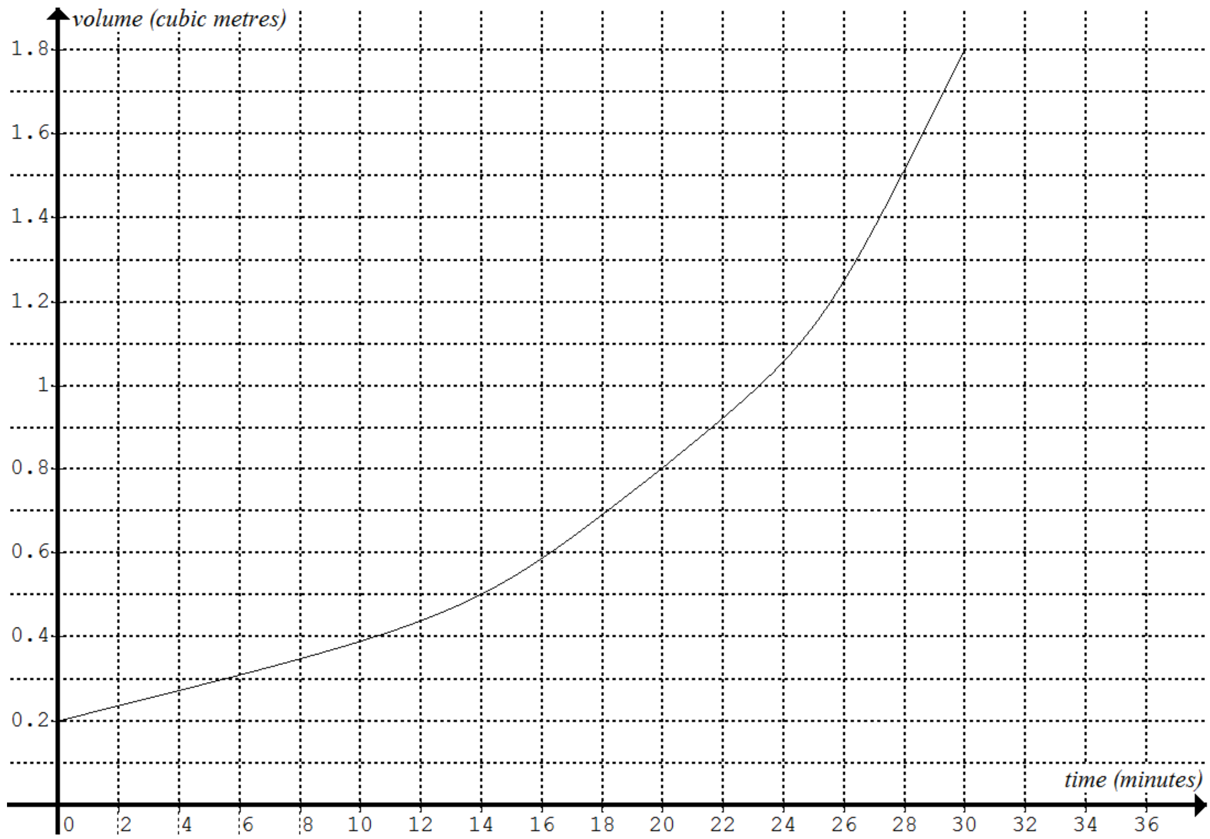
After transformations: $y = a(x+b)^n$

Choose $n = 5$ and the two end points to find the values of a and b, correct to four significant figures.

3 marks

h. Repeat Part I g choosing $n = 10, 20$ and 30. 6 marks

i. Sketch the functions found in Part I g and h clearly in the following diagram if possible. 8 marks



j. Discuss the trend and suggest a much greater n value which may give a better fit to the original curve. Use your suggested n value to determine the values of a and b. Comment on any unexpected occurrences and the limitations of your calculator.

5 marks

Part II (40 marks)

The shape of the original curve suggests that an exponential function with appropriate transformations could better fit the curve.

Before transformations: $y = 2^x$

After transformations: $y = a2^{bx} + c$

a. State, in correct order, the transformations of $y = 2^x$ to $y = a2^{bx} + c$ represented by the parameters a, b and c.

4 marks

b. Three pieces of information are required to evaluate the three parameters a, b and c.
Use the two end points and the point $(20, 0.8)$ on the curve to set up three simultaneous equations involving parameters a, b and c.

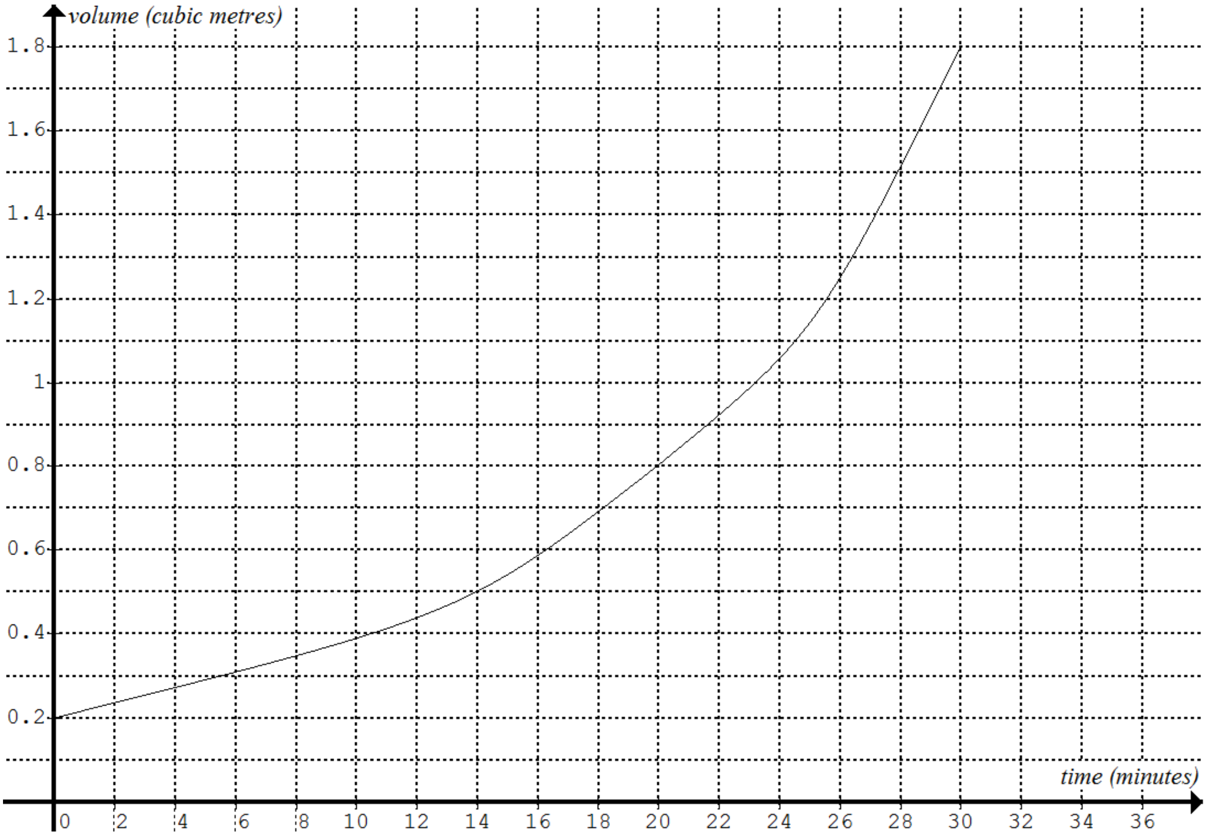
3 marks

c. Show that $\dfrac{(0.8-c)^3}{(0.2-c)(1.8-c)^2} = 1$ algebraically.

5 marks

d. Hence show that $a \approx 0.13$, $b \approx 0.12447$ and $c \approx 0.07$. 4 marks

e. Sketch the graph of $y = a2^{bx} + c$ on the diagram below. 3 marks



f. $y = a2^{bx} + c$ can be expressed in the form $y = \alpha e^{\beta x} + \gamma$, where $a \approx 0.13$, $b \approx 0.12447$ and $c \approx 0.07$. Find the value of each of α, β and γ.

3 marks

g. Using $y = a2^{bx} + c$ as an approximation to the original curve, find an approximate value of the gradient of the tangent to the original curve at $(20, 0.8)$. Hence find the equation of the tangent.

4 marks

h. Using $y = a2^{bx} + c$ as an approximation to the original curve, find approximate value of the gradient of the tangent to the original curve at each of the two endpoints. Hence find the equation of each tangent.

5 marks

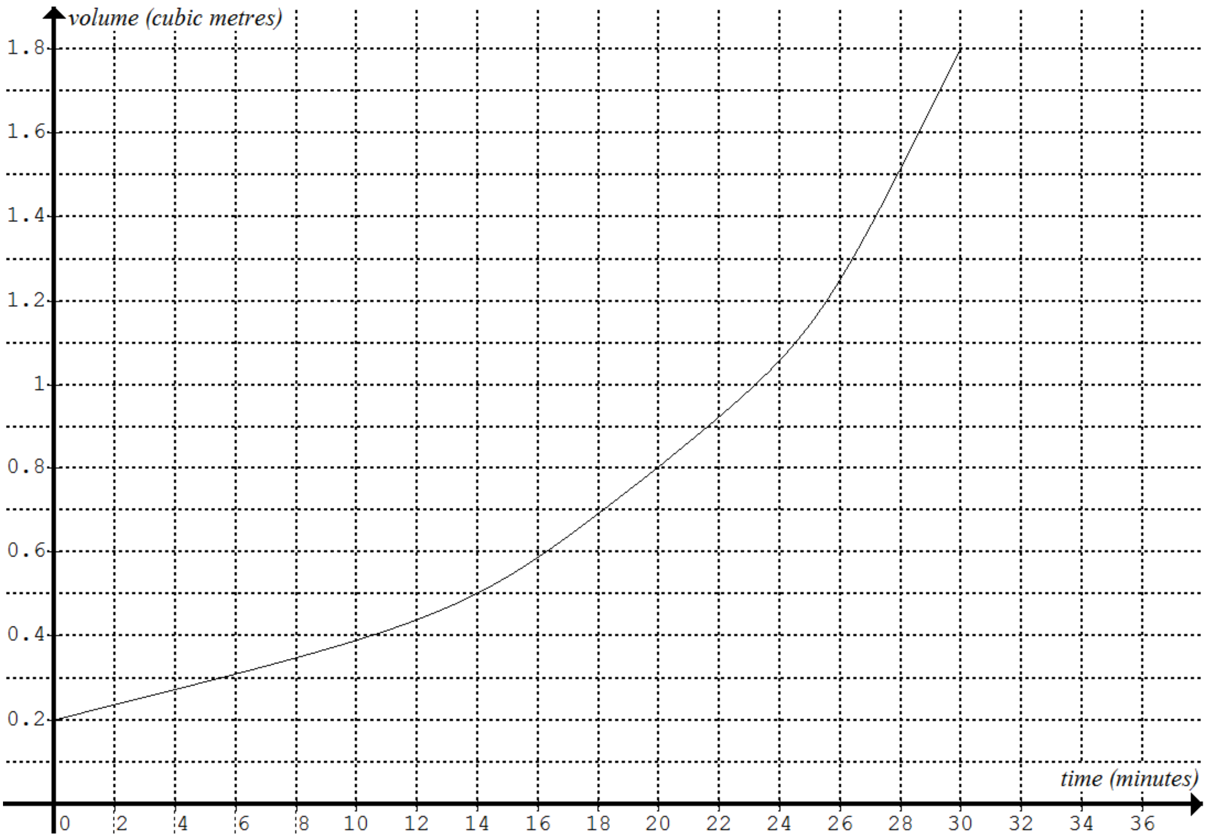
i. Sketch the tangents at $(20, 0.8)$ and the endpoints on the graph shown in Part II e, satisfying the equations of the tangents.

6 marks

j. Comment on the the goodness of fit of the tangents to the original curve. 3 marks

Part III (35 marks)

The original curve is shown below.



A close study of the original curve identifies three approximately linear sections in the following intervals:

$$0 \le t < 10\,,\ 18 \le t < 22\,,\ 26 \le t \le 30$$

The coordinates of the endpoints are $(0, 1.2)$, $(10, 0.385)$, $(18, 0.69)$, $(22, 0.925)$, $(26, 1.25)$ and $(30, 1.8)$.

a. Assuming the three sections are linear, determine the equation of each section. 6 marks
Check your answers before you move to part b.

b. The section in the interval $22 \le t < 26$ is a curve smoothly joined to its adjacent neighbours.
Note: Two curves are smoothly joined if they have the same gradient at the joining point.

A suitable curve is a polynomial function of degree three in the form $V(t) = at^3 + bt^2 + ct + d$.
Use coordinates of points and gradient at the points to form four linear equations involving a, b, c and d.

10 marks

c. Determine the value of each of a, b, c and d. 4 marks

d. Explain why the curve cannot be part of a parabola. 3 marks

e. What further information is required if a polynomial of degree four is used to model the curve in the interval $10 \le t < 18$?

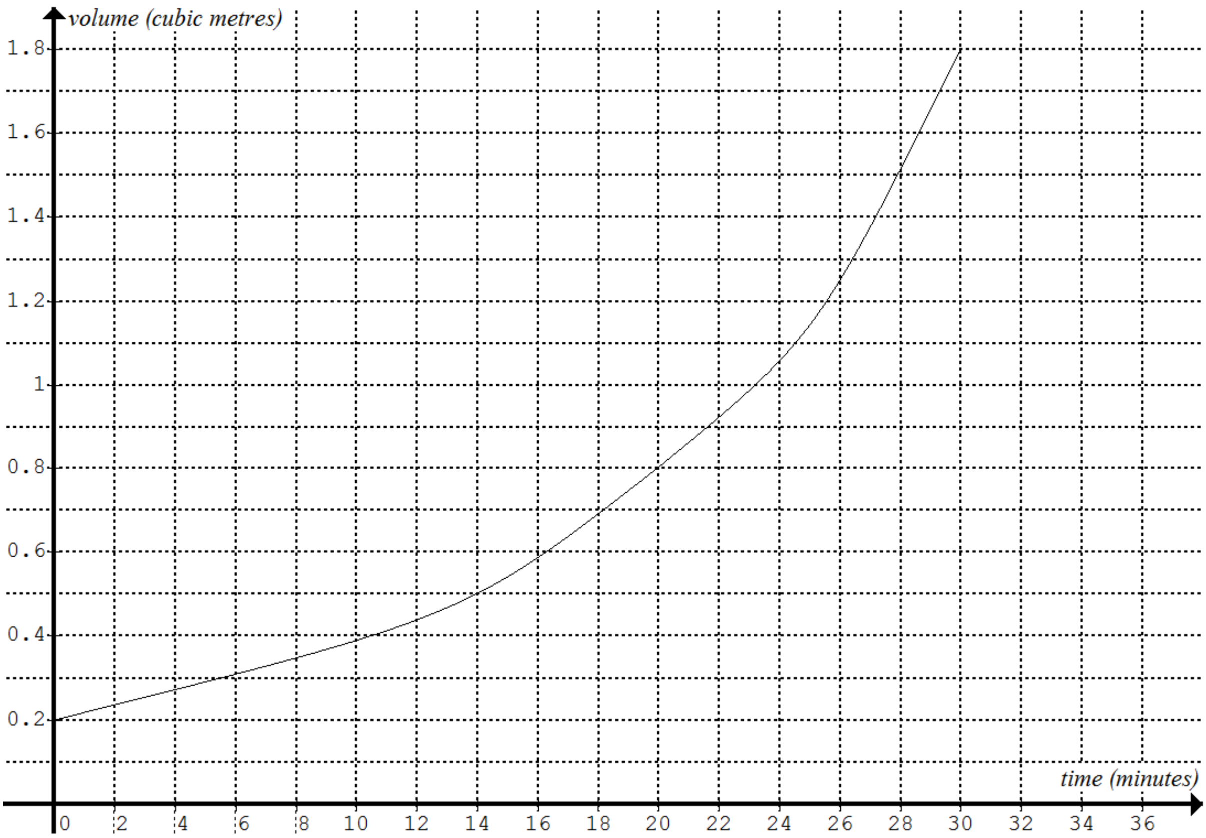
2 marks

f. In the interval $10 \le t < 18$, use a polynomial of a degree **higher than three** to model the curve joined smoothly to its adjacent neighbours.

5 marks

g. Summarise the five sections as a hybrid function, showing the corresponding intervals. 2 marks

h. Sketch in the following graph the hybrid function in Part III g. 3 marks



End of task