2018 NSW ESA Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
B	A	A	D	A	C	C	B	D	B

Q1 $\alpha\beta\gamma = 5$, $\alpha+\beta+\gamma = -3$, $\alpha\beta\gamma(\alpha+\beta+\gamma) = -15$ **B**

Q2 $\theta = \tan^{-1}5 - \tan^{-1}3$, $\tan\theta = \frac{5-3}{1+5\times 3} = \frac{1}{8}$ **A**

Q3 $\lim_{x\to 0}\frac{1}{4}\times\frac{\sin 6x}{6x} = \frac{1}{4}$ **A**

Q4 $-6 = a(2)(1)(-1)^2$, $a = -3$ **D**

Q5 Asymptote: $P = 1500$, $P(0) = 1500 + 1500e^0 = 3000$ **A**

Q6 $f'(c) \approx \frac{0 - f(c)}{w - c}$ **C**

Q7 $a = v\frac{dv}{dx} = (x^2+2)(2x) = 6$ **C**

Q8 A person is seated first, the remaining 5 of the same sex can be arranged in 5! ways sitting one chair apart.
For each of these arrangements, the 6 of the opposite sex can now be seated in 6! ways. .: 5! 6! ways **B**

Q9 $\sin 2x = -\frac{1}{2}$

$2x = \sin^{-1}\left(-\frac{1}{2}\right) = n\pi + (-1)^n\left(-\frac{\pi}{6}\right) = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right)$

.: $x = \frac{n\pi}{2} + (-1)^{n+1}\frac{\pi}{12}$ **D**

Q10 $v = 0$, $x = 0, 2k$, .: centre is at $x = k$ and amplitude is k.
Also when $t = 0$, $x = k$. **B**

Section II

Q11ai $P(1) = 1^3 - 2(1^2) - 5(1) + 6 = 0$

Q11aii $P(x) = (x-1)(x^2 - x - 6) = (x-1)(x+2)(x-3)$, $x = -2, 3$

Q11b $\log_2 5(x-2) = 3$, $5(x-2) = 8$, $x = \frac{18}{5}$

Q11c $R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$, .: $R^2 = 4$, $R = 2$ and $\alpha = \frac{\pi}{6}$

Q11d $x(x+2) = 3\times 8$, $x(x+2) = 4\times 6$, .: $x = 4$

Q11ei $4x - 1 \neq 0$, $x \neq \frac{1}{4}$, domain: $R\setminus\left\{\frac{1}{4}\right\}$

Q11eii For $4x-1>0$ and $\frac{1}{4x-1} < 1$, $4x - 1 > 1$, $x > \frac{1}{2}$

Also $\frac{1}{4x-1} < 1$ when $4x - 1 < 0$, .: $x < \frac{1}{4}$ or $x > \frac{1}{2}$

Q11f $u = 1 - x$, $x = 1 - u$

$$\int_{-3}^{0}\frac{x}{\sqrt{1-x}}dx = \int_{4}^{1}\frac{1-u}{\sqrt{u}}(-du) = \int_{1}^{4}\left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)du = \left[2u^{\frac{1}{2}} - \frac{2u^{\frac{3}{2}}}{3}\right]_1^4 = -\frac{8}{3}$$

Q12a $\int\cos^2(3x)dx = \frac{1}{2}\int(1+\cos(6x))dx = \frac{1}{2}x + \frac{1}{12}\sin(6x) + C$

Q12bi $h = 20\sin\theta$, $\frac{dh}{d\theta} = 20\cos\theta$, when $h = 15$, $\theta = \sin^{-1}\frac{15}{20}$

Vertical speed $= \frac{dh}{dt} = \frac{dh}{d\theta}\times\frac{d\theta}{dt}$

$= 20\cos\left(\sin^{-1}\frac{15}{20}\right)\times 1.5 \approx 19.8$ m per minute

Q12ci $f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$

Q12cii .: $f(x) =$ constant for $-1 < x < 1$

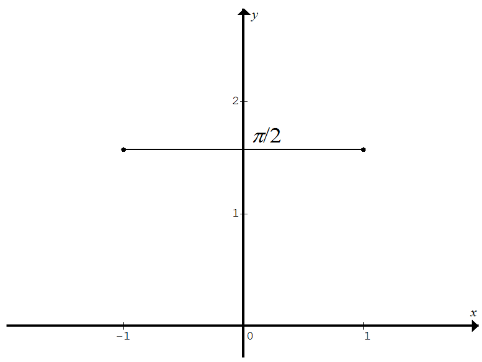
Let $x = 0$, $f(0) = \sin^{-1}0 + \cos^{-1}0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$,

.: $f(x) = \frac{\pi}{2}$ for $-1 < x < 1$

For $x = -1$ or 1, $f(-1) = \sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$

$f(1) = \sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$, .: $f(x) = \frac{\pi}{2}$ for $-1 \le x \le 1$

Q12ciii



Q12d $\Pr(X \ge 10) = \Pr(X = 10) + \Pr(X = 11) + \Pr(X = 12)$

$= \binom{12}{10}0.75^{10}\,0.25^2 + \binom{12}{11}0.75^{11}\,0.25 + \binom{12}{12}0.75^{12}$

Q12ei $T(a(p+q), apq)$ is true for all q.
Let $q = 0$, .: $T(ap, 0) = A(ap, 0)$

$m_{SA} = \frac{a-0}{0-ap} = -\frac{1}{p}$ and $m_{PA} = \frac{ap^2 - 0}{2ap - ap} = p$.: $m_{SA}\times m_{PA} = -1$

.: $\angle PAS = 90°$

Q12eii $T(a(p+q), apq)$ is true for all p.

Let $p=0$, .: $T(aq, 0) = B(aq, 0)$

$m_{SB} = \frac{a-0}{0-aq} = -\frac{1}{q}$ and $m_{QB} = \frac{aq^2-0}{2aq-aq} = q$.: $m_{SB} \times m_{QB} = -1$

.: $\angle QBS = 90°$

Since $\angle PAS = \angle TAS = 90°$ and $\angle QBS = \angle TBS = 90°$

.: ST is a diameter of a circle through S, B, A and T.

Q12eiii $ST = \sqrt{(a(p+q)-0)^2 + (apq-a)^2}$

$= \sqrt{a^2(p^2+2pq+q^2) + a^2(p^2q^2-2pq+1)}$

$= a\sqrt{p^2q^2+p^2+q^2+1} = a\sqrt{(p^2+1)(q^2+1)}$ since $a>0$

Q13a For $n=1$, $LS = 2(-3)^0 = 2$, $RS = \frac{1-(-3)^1}{2} = 2$, .: true

Assume it is true for $n=k$,

.: $2-6+18-54+...+2(-3)^{k-1} = \frac{1-(-3)^k}{2}$

Consider $n=k+1$,

$LS = 2-6+18-54+...+2(-3)^{k-1}+2(-3)^k$

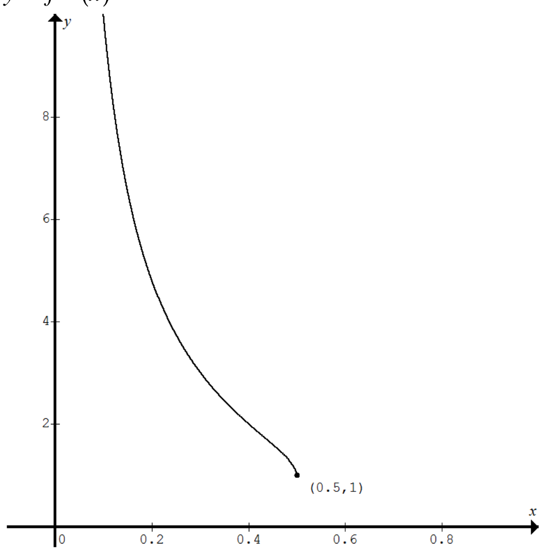
$= \frac{1-(-3)^k}{2} + 2(-3)^k = \frac{1-(-3)^k+4(-3)^k}{2}$

$= \frac{1+3(-3)^k}{2} = \frac{1-(-3)(-3)^k}{2} = \frac{1-(-3)^{k+1}}{2} = RS$

.: it is true for $n=k+1$. Hence true for all $n \ge 1$.

Q13bi $f^{-1}(x)$, domain: $\left(0, \frac{1}{2}\right]$, range: $[1, \infty)$

Q13bii $y = f^{-1}(x)$



Q13biii $x = \frac{y}{y^2+1}$, $xy^2 - y + x = 0$, $y = \frac{1+\sqrt{1-4x^2}}{2x}$

.: $f^{-1}(x) = \frac{1+\sqrt{1-4x^2}}{2x}$

Q13ci Let $y = Vt\sin\theta - \frac{gt^2}{2} = 0$, $t=0$ or $\frac{2V\sin\theta}{g}$ when the object is at the same level. .: $x=0$ or $x = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta = \frac{V^2}{g}\sin 2\theta$

.: the horizontal range is $\frac{V^2}{g}\sin 2\theta$.

Q13cii The horizontal range is

$\frac{V^2}{g}\sin 2\left(\frac{\pi}{2}-\theta\right) = \frac{V^2}{g}\sin(\pi-2\theta) = \frac{V^2}{g}\sin 2\theta$

Q13ciii Maximum range at $\theta = \frac{\pi}{4}$, .: $\alpha \ne \beta \ne \frac{\pi}{4}$

Max height occurs at half of the flight time, .: $t_\alpha = \frac{V\sin\alpha}{g}$

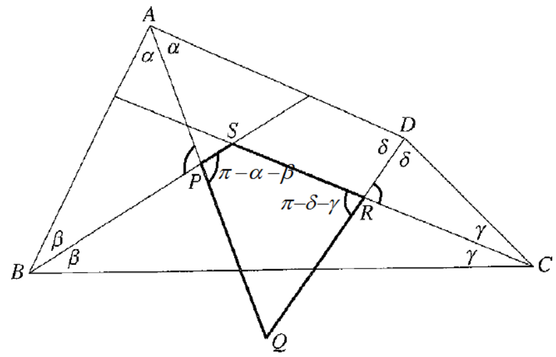
$h_\alpha = V\frac{V\sin\alpha}{g}\sin\alpha - \frac{g}{2}\left(\frac{V\sin\alpha}{g}\right)^2 = \frac{V^2\sin^2\alpha}{2g}$.

Similarly, $h_\beta = \frac{V^2\sin^2\beta}{2g}$.

$\frac{h_\alpha + h_\beta}{2} = \frac{V^2}{4g}(\sin^2\alpha + \sin^2\beta)$

$= \frac{V^2}{4g}\left(\sin^2\alpha + \sin^2\left(\frac{\pi}{2}-\alpha\right)\right) = \frac{V^2}{4g}(\sin^2\alpha + \cos^2\alpha) = \frac{V^2}{4g}$

Q14ai



$ABCD$ is a quadrilateral,

$2\alpha + 2\beta + 2\gamma + 2\delta = 2\pi$.: $\alpha+\beta+\gamma+\delta = \pi$

$\angle SPQ + \angle SRQ = 2\pi - (\alpha+\beta+\gamma+\delta) = \pi$

.: $PQRS$ is a cyclic quadrilateral.

Q14bi Compare the coefficients of x^r in the two expansions.

$(2+x)^n = ...+\binom{n}{r}2^{n-r}x^r+......$

$(1+(1+x))^n = ...+\binom{n}{r}(1+x)^r+\binom{n}{r+1}(1+x)^{r+1}$

$+\binom{n}{r+2}(1+x)^{r+2}+...+\binom{n}{n}(1+x)^n$

$=...+\binom{n}{r}\left(...+\binom{r}{r}x^r\right)+\binom{n}{r+1}\left(...+\binom{r+1}{r}x^r+...\right)$

$+\binom{n}{r+2}\left(...+\binom{r+2}{r}x^r+...\right)+...+\binom{n}{n}\left(...+\binom{n}{r}x^r+...\right)$

$\therefore \binom{n}{r}2^{n-r}=\binom{n}{r}\binom{r}{r}+\binom{n}{r+1}\binom{r+1}{r}+\binom{n}{r+2}\binom{r+2}{r}+...+\binom{n}{n}\binom{n}{r}$

Q14bii Select 4 from 23 and then 4 from 4: $\binom{23}{4}\binom{4}{4}$

Select 5 from 23 and then 4 from 5: $\binom{23}{5}\binom{5}{4}$

Select 6 from 23 and then 4 from 6: $\binom{23}{6}\binom{6}{4}$

……

Select 23 from 23 and then 4 from 23: $\binom{23}{23}\binom{23}{4}$

Total number of ways $=\binom{23}{4}2^{23-4}=\binom{23}{4}2^{19}$ from part i.

Q14ci ΔABC is similar to ΔACD because they have common $\angle A$, $\angle ACB=\angle ADC=90°$, then $\angle ABC=\angle ACD$. AAA

Q14cii $\therefore \frac{x}{b}=\frac{a}{c}$, $x=\frac{ab}{c}$

Q14ciii $\sin\angle A=\frac{x}{b}=\frac{x_1}{b-x}=\frac{x_2}{b-x-x_1}=\frac{x_3}{b-x-x_1-x_2}=...$

Since $x=\frac{ab}{c}$, $x_1=\left(1-\frac{a}{c}\right)x$, $x_2=\left(1-\frac{a}{c}\right)^2x$, $x_3=\left(1-\frac{a}{c}\right)^3x$, …

Limiting sum $=\frac{\pi}{4}\left(x^2+x_1^{\ 2}+x_2^{\ 2}+x_3^{\ 2}+...\right)$

$=\frac{\pi}{4}x^2\left(1+\left(1-\frac{a}{c}\right)^2+\left(1-\frac{a}{c}\right)^4+\left(1-\frac{a}{c}\right)^6+...\right)$ Note: $0<\frac{a}{c}<1$

$=\frac{\pi}{4}\times\frac{a^2b^2}{c^2}\times\frac{1}{1-\left(1-\frac{a}{c}\right)^2}=\frac{\pi}{4}\times\frac{a^2b^2}{c^2}\times\frac{1}{\frac{2a}{c}-\frac{a^2}{c^2}}=\frac{\pi ab^2}{4(2c-a)}$

Q14civ $\frac{\pi ab^2}{4(2c-a)}<\frac{1}{2}ab$, $\therefore \frac{\pi}{2}<\frac{2c-a}{b}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.