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2018 NSW ESA Mathematics Extension 2 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
B	C	D	C	A	A	D	B	B	C

Q1 $\int \frac{1}{\sqrt{1-(2x)^2}}dx = \frac{1}{2}\sin^{-1} 2x + C$ B

Q2 $\frac{x^2}{4} - \frac{y^2}{9} = 1$, asymptotes: $y = \pm\frac{3}{2}x$ C

Q3 $\alpha\beta\gamma = 1$, $\alpha\beta + \beta\gamma + \gamma\alpha = 5$, $\alpha + \beta + \gamma = -2$

$(\alpha x + 1)(\beta x + 1)(\gamma x + 1)$

$= \alpha\beta\gamma\, x^3 + (\alpha\beta + \beta\gamma + \gamma\alpha)x^2 + (\alpha + \beta + \gamma)x + 1$

$= x^3 + 5x^2 - 2x + 1 = 0$ D

Q4 As $x \to \pm\infty$, $y \to 0^+$. As $x \to \pm 2$, $y \to +\infty$. C

Q5 A

Q6 $z^6 = \text{cis}\left(\frac{\pi}{2} + 2n\pi\right)$, $z = \text{cis}\frac{(1+4n)\pi}{12}$

When $n = 2$, $z = \text{cis}\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ A

Q7 For z and $z + 1 - i$ to have the same argument, they must be in the same quadrant. .: $z \in$ A or D. D is better than A because it covers two quadrants, 2nd and 4th. D

Q8 B

$F''(x) = f'(x) > 0$ for $0 < x < a$, $F(x)$ concave upward

$F''(x) = f'(x) < 0$ for $a < x < c$, $F(x)$ concave downward

$F''(x) = f'(x) > 0$ for $c < x < d$, $F(x)$ concave upward

$F''(x) = f'(x) > 0$ for $d < x$, $F(x)$ concave upward

Concavity changes at $x = a$, c.

Q9 ab, pq and $ab - pq$ are real, and ap, bq and $ap - bq$ are purely imaginary

$(ab - pq)^2 \geq 0$, .: $a^2b^2 + p^2q^2 \geq 2abpq$

$(ap - bq)^2$ is a negative real number, $(ap - bq)^2 \leq 0$

.: $a^2p^2 + b^2q^2 \leq 2abpq$ B

Q10 Let $g'(x) = x\cos x + \sin x = 0$, .: $\tan x = -x$, $\tan b = -b$

Stationary points of $f(x)$ occur at the asymptotes of $y = \tan x$.

For $x > 0$, $\tan b = -b$ occurs to the right of asymptote $x = a$.

For $x < 0$, $\tan b = -b$ occurs to the left of asymptote $x = a$.

.: $|a| < |b|$ C

Section II

Q11ai $zw = (2 + 3i)(1 - i) = 5 + i$

Q11aii $\bar{z} - \frac{2}{w} = 2 - 3i - \frac{2(1+i)}{(1-i)(1+i)} = 1 - 4i$

Q11b $4^2 r = -b$, $4^2 + 4r + 4r = 0$, $4 + 4 + r = -a$

.: $r = -2$, $b = 32$, $a = -6$

Q11c $a(x^2 - 3) + (x+1)(bx + c) = x^2 - x - 6$

$a + b = 1$, $-3a + c = -6$, $b + c = -1$, .: $a = 2$, $b = -1$, $c = 0$

$\int\left(\frac{2}{x+1} - \frac{x}{x^2 - 3}\right)dx = 2\log_e|x+1| - \frac{1}{2}\log_e|x^2 - 3| + C$

$= \log_e\frac{(x+1)^2}{\sqrt{|x^2 - 3|}} + C$

Q11di $w = iu = -2 + 5i$

Q11dii $v = u + w = 3 + 7i$

Q11diii $\arg\left(\frac{w}{v}\right) = \arg(w) - \arg(v) = \frac{\pi}{4}$

Q11e $\angle CBA = \angle CDA$ subtended by the same chord.

AB is a diameter, .: ΔCBA is a right-angle triangle.

.: $d = 2r\sin\angle CBA = 2r\sin\angle CDA = 2r\sin D$

Q12ai $V = 2\int_0^1 \frac{1}{2}x^2 \sin 60°\, dy = \frac{\sqrt{3}}{2}\int_0^1 (1 - y^2)^2 dy$

$= \frac{\sqrt{3}}{2}\int_0^1 (1 - 2y^2 + y^4)dy = \frac{\sqrt{3}}{2}\left[y - \frac{2y^3}{3} + \frac{y^5}{5}\right]_0^1 = \frac{4\sqrt{3}}{15}$

Q12bi $x^2 + xy + y^2 = 3$, $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$

Q12bii $\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = 0$, $2x + y = 0$ and $x^2 + xy + y^2 = 3$

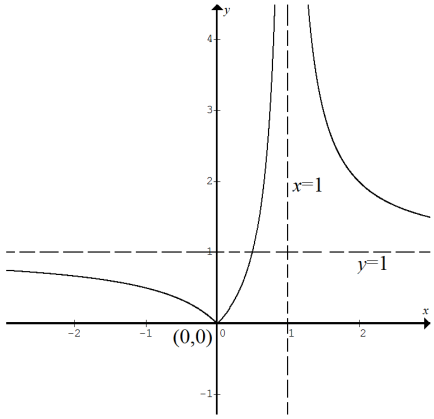
.: $x^2 + x(-2x) + (-2x)^2 = 3$, $3x^2 = 3$, $x = \pm 1$ and $y = \mp 2$

$(1, -2)$ and $(-1, 2)$

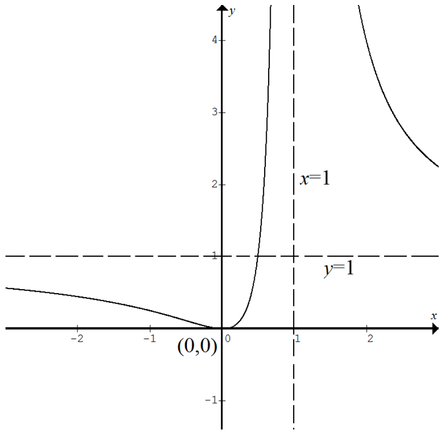
Q12c

$\int\frac{x^2 + 2x}{x^2 + 2x + 5}dx = \int\left(1 - \frac{5}{(x+1)^2 + 4}\right)dx = x - \frac{5}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

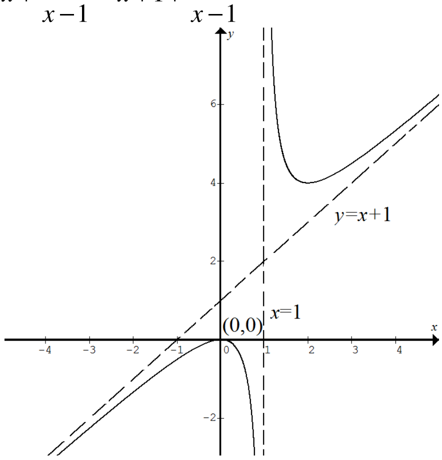
Q12di



Q12dii



Q12diii $y = x + \frac{x}{x-1} = x + 1 + \frac{1}{x-1}$



Q13a At $0 \le x \le 1$, radius $= 1 - x$, $V = 2\int_0^1 2\pi(1-x)h\,dx$

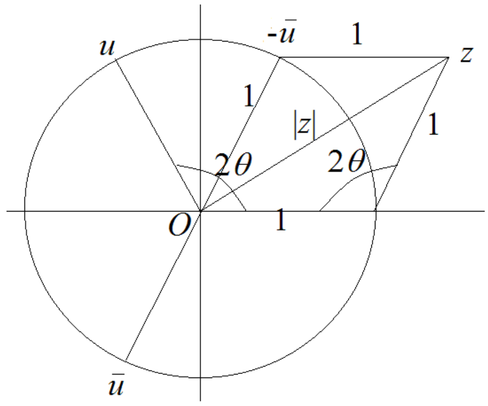
$= 2\int_0^1 2\pi(1-x)(1-x)\sqrt{x}\,dx = 4\pi\int_0^1 (1-2x+x^2)x^{\frac{1}{2}}\,dx$

$= 4\pi\left[\frac{2x^{\frac{3}{2}}}{3} - \frac{4x^{\frac{5}{2}}}{5} + \frac{2x^{\frac{7}{2}}}{7}\right]_0^1 = \frac{64\pi}{105}$

Q13bi $z = 1 - \cos 2\theta + i\sin 2\theta = 1 - (\cos 2\theta - i\sin 2\theta)$

$= 1 - \left(\overline{\cos 2\theta + i\sin 2\theta}\right)$

Let $u = \cos 2\theta + i\sin 2\theta$, $z = 1 + (-\bar{u})$



$|z|^2 = 1^2 + 1^2 - 2(1)(1)\cos 2\theta = 2 - 2\cos 2\theta = 2 - 2(1 - 2\sin^2\theta)$

$|z| = 2\sin\theta$

Q13bii Consider isosceles $\Delta Oz1$, $\arg(z) = \frac{\pi - 2\theta}{2} = \frac{\pi}{2} - \theta$

Q13c Vertical: $N + T\cos\theta - mg = 0$

Horizontal: $T\sin\theta = mr\omega^2$

Solve simultaneously, $\cot\theta = \frac{mg - N}{mr\omega^2}$

$N = m(g - r\omega^2\cot\theta) \ge 0$ for the particle to stay in contact with the horizontal surface.

.: $g - r\omega^2\cot\theta \ge 0$

Since $r = \ell\sin\theta$, .: $\omega^2 \le \frac{g}{\ell\cos\theta}$

Q13di Given $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$

and R is $(c(p+q), 0)$ and S is $\left(0, \frac{c(p+q)}{pq}\right)$

Using Pythagoras theorem, it can be shown $PS^2 = c^2p^2 + \frac{c^2}{q^2}$ and

$QR^2 = c^2p^2 + \frac{c^2}{q^2}$, .: $PS = QR$

Q13dii Let $C(at, 0)$ and $D(0, -at^2)$ be the x and y intercepts of AB respectively.

From part (i), $PS = QR$, .: the midpoints of PQ and RS are the same point.

.: the midpoints of AB and CD are the same point M.

.: $M(x, y)$ where $x = \frac{at}{2}$, and $y = -\frac{at^2}{2}$

.: the locus of M is $2x^2 = -ay$ by eliminating t.

Q14ai $t=\tan\frac{\theta}{2}$, $\theta=2\tan^{-1}t$, $\frac{d\theta}{dt}=\frac{2}{1+t^2}$

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{2-\cos\theta}=\int_0^1\frac{2dt}{(1+t^2)\left(2-\frac{1-t^2}{1+t^2}\right)}=2\int_0^1\frac{dt}{1+3t^2}=2\left[\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3}\,t\right]_0^1$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{3}=\frac{2\sqrt{3}\,\pi}{9}$$

Q14b

Let V be the speed after falling from rest through a distance, h.

$\frac{1}{2}\frac{d(v^2)}{dx}=g-k(v^2)$, $2\frac{dx}{d(v^2)}=\frac{\frac{1}{k}}{\frac{g}{k}-(v^2)}$,

$$2kh=\int_0^{(v^2)}\frac{1}{\frac{g}{k}-(v^2)}d(v^2)=\left[-\log_e\left(\frac{g}{k}-(v^2)\right)\right]_0^{(v^2)}=-\log_e\left(\frac{\frac{g}{k}-V^2}{\frac{g}{k}}\right)$$

.: $2kh=-\log_e\left(1-\frac{kV^2}{g}\right)$, $1-\frac{kV^2}{g}=e^{-2kh}$, $V=\sqrt{\frac{g}{k}\left(1-e^{-2kh}\right)}$

Q14c Let $u=x^n$ and $\frac{dv}{dx}=\sqrt{x+3}$, .: $\frac{du}{dx}=nx^{n-1}$ and $v=\frac{2}{3}(x+3)^{\frac{3}{2}}$

$$\int_{-3}^{0}\frac{d}{dx}(uv)dx=\int_{-3}^{0}u\frac{dv}{dx}dx+\int_{-3}^{0}v\frac{du}{dx}dx$$

$$\left[x^n\frac{2}{3}(x+3)^{\frac{3}{2}}\right]_{-3}^{0}=I_n+\int_{-3}^{0}\frac{2}{3}(x+3)^{\frac{3}{2}}nx^{n-1}dx$$

$$0=I_n+\frac{2n}{3}\int_{-3}^{0}(x+3)\sqrt{x+3}\,x^{n-1}dx$$

$$0=I_n+\frac{2n}{3}\left(\int_{-3}^{0}x^n\sqrt{x+3}\,dx+\int_{-3}^{0}3x^{n-1}\sqrt{x+3}\,dx\right)$$

$$0=I_n+\frac{2n}{3}(I_n+3I_{n-1}),\ 0=3I_n+2nI_n+6nI_{n-1}$$

.: $I_n=\frac{-6n}{3+2n}I_{n-1}$

Q14cii $I_0=\int_{-3}^{0}\sqrt{x+3}\,dx=\left[\frac{2(x+3)^{\frac{3}{2}}}{3}\right]_{-3}^{0}=2\sqrt{3}$

$I_1=\frac{-6}{5}I_0=\frac{-12\sqrt{3}}{5}$, $I_2=\frac{-12}{7}I_1=\frac{144\sqrt{3}}{35}$

Q14di n independent games, $\left(\frac{1}{3}\right)^n$

Q14dii (Probability that C never wins) minus (probability that A wins every game) minus (probability that B wins every game)

$=\left(\frac{2}{3}\right)^n-2\left(\frac{1}{3}\right)^n$

Q14diii $1-3\times(\text{i})-3\times(\text{ii})=1-3\left(\frac{1}{3}\right)^n-3\left(\left(\frac{2}{3}\right)^n-2\left(\frac{1}{3}\right)^n\right)$

$=1+3\left(\frac{1}{3}\right)^n-3\left(\frac{2}{3}\right)^n=1+\frac{1}{3^{n-1}}-\frac{2^n}{3^{n-1}}=\frac{3^{n-1}-2^n+1}{3^{n-1}}$

Q15ai x-coord. of $Q=x$-coord. of $B=a\cos\left(\theta+\frac{\pi}{2}\right)=-a\sin\theta$

$\frac{(-a\sin\theta)^2}{a^2}+\frac{y^2}{b^2}=1$, $y=\pm b\cos\theta$

.: $Q(-a\sin\theta,b\cos\theta)$

Q15aii Let $\phi=\angle POQ'=\tan^{-1}\left(\frac{b\sin\theta}{a\cos\theta}\right)-\tan^{-1}\left(\frac{-b\cos\theta}{a\sin\theta}\right)$

$$\tan\phi=\frac{\frac{b\sin\theta}{a\cos\theta}-\left(\frac{-b\cos\theta}{a\sin\theta}\right)}{1+\left(\frac{b\sin\theta}{a\cos\theta}\right)\left(\frac{-b\cos\theta}{a\sin\theta}\right)}=\frac{2ab}{(a^2-b^2)\sin2\theta}$$

ϕ and $\tan\phi$ are minimum when $\sin2\theta$ is maximum, i.e. 1

$\tan\phi=\frac{2ab}{a^2-b^2}$, .: max $\phi=\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$

Q15bi

$(\cos\theta+i\sin\theta)^8=\cos8\theta+i\sin8\theta$

$$(\cos\theta+i\sin\theta)^8=\cos^8\theta+\binom{8}{1}\cos^7\theta\,i\sin\theta+\binom{8}{2}\cos^6\theta(i\sin\theta)^2$$
$$+\binom{8}{3}\cos^5\theta(i\sin\theta)^3+\binom{8}{4}\cos^4\theta(i\sin\theta)^4$$
$$+\binom{8}{5}\cos^3\theta(i\sin\theta)^5+\binom{8}{6}\cos^2\theta(i\sin\theta)^6$$
$$+\binom{8}{7}\cos\theta(i\sin\theta)^7+\binom{8}{8}(i\sin\theta)^8$$

.:

$$i\sin8\theta=\binom{8}{1}\cos^7\theta\,i\sin\theta+\binom{8}{3}\cos^5\theta(i\sin\theta)^3+\binom{8}{5}\cos^3\theta(i\sin\theta)^5$$
$$+\binom{8}{7}\cos\theta(i\sin\theta)^7$$

$$\sin8\theta=\binom{8}{1}\cos^7\theta\sin\theta-\binom{8}{3}\cos^5\theta\sin^3\theta+\binom{8}{5}\cos^3\theta\sin^5\theta$$
$$-\binom{8}{7}\cos\theta\sin^7\theta$$

Q15bii $\sin8\theta=4\cos^6\theta\sin2\theta-28\cos^4\theta\sin^2\theta\sin2\theta$

$+28\cos^2\theta\sin^4\theta\sin2\theta-4\sin^6\theta\sin2\theta$

$\frac{\sin8\theta}{\sin2\theta}=4\cos^6\theta-28\cos^4\theta\sin^2\theta+28\cos^2\theta\sin^4\theta-4\sin^6\theta$

$=4(1-\sin^2\theta)^3-28(1-\sin^2\theta)^2\sin^2\theta+28(1-\sin^2\theta)\sin^4\theta$

$-4\sin^6\theta$

$=4(1-10\sin^2\theta+24\sin^4\theta-16\sin^6\theta)$

Q15ci $x^n-1-n(x-1)=(x-1)(x^{n-1}+x^{n-2}+\ldots+x^2+x+1)-n(x-1)$
$=(x-1)(1+x+x^2+\ldots+x^{n-1}-n)$

Q15cii Consider $P(x)=(x-1)(1+x+x^2+\ldots+x^{n-1}-n)$
$P(1)=0$, $P(0<x<1)>0$, $P(x>1)>0$
.: $P(x)\geq 0$ for $x\in R^+$
.: $x^n-1-n(x-1)\geq 0$, $x^n\geq 1+n(x-1)$

Q15ciii $x^n\geq 1+n(x-1)$, $\left(\frac{a}{b}\right)^n\geq 1+n\left(\frac{a}{b}-1\right)$,
$\left(\frac{a}{b}\right)^n b\geq\left(1+n\left(\frac{a}{b}-1\right)\right)b$, .: $a^nb^{1-n}\geq na+(1-n)b$

Q16a $n=1$, $x^3-1=(x-1)(x^2+x+1)$ is true
$n=k$, $x^{(3^k)}-1=(x-1)(x^2+x+1)(x^6+x^3+1)\ldots(x^{(2\times3^{k-1})}+x^{(3^{k-1})}+1)$
$n=k+1$,
$x^{(3^{k+1})}-1=x^{(3\times3^k)}-1=\left(x^{(3^k)}\right)^3-1=\left(x^{(3^k)}-1\right)\left(x^{(2\times3^k)}+x^{(3^k)}+1\right)$
$=(x-1)(x^2+x+1)(x^6+x^3+1)\ldots\left(x^{(2\times3^k)}+x^{(3^k)}+1\right)$
.: true for $n\geq 1$

Q16bi ΔGBF and ΔIHC are similar to ΔABC.
The side length of ΔGBF is $\frac{1}{\sqrt{2}}$ of the corresponding side length of ΔABC, the side length of ΔIHC is also $\frac{1}{\sqrt{2}}$ of the corresponding side length of ΔABC, .: ΔGBF and ΔIHC are congruent.
Consider ΔIHC as the translation of ΔGBF to the right.
.: $DY=ZE$.

Q16bii Similar triangles, if $\frac{BC}{DE}=\sqrt{2}$, $\frac{BC}{BF}=\sqrt{2}$, .: $DE=BF$
.: $DEFB$ and .: $YEFH$ are parallelograms, .: $YE=HF$
Let $ZE=1$ unit, $HF=YZ+1$
.: $\frac{BC}{DE}=\frac{3+YZ}{2+YZ}=\sqrt{2}$, $YZ=\sqrt{2}-1$
$\frac{YZ}{BC}=\frac{\sqrt{2}-1}{2+HF}=\frac{\sqrt{2}-1}{2+\sqrt{2}}=\frac{3\sqrt{2}}{2}-2$

Q16ci $\alpha\beta\gamma=-q$, $\alpha\beta+\beta\gamma+\gamma\alpha=p$, $\alpha+\beta+\gamma=0$
$p(\alpha)=\alpha^3+p\alpha+q=0$, $p=\frac{-\alpha^3-q}{\alpha}$
$(\beta-\gamma)^2=\beta^2+\gamma^2-2\beta\gamma=(\alpha+\beta+\gamma)^2-\alpha^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)-\frac{2\alpha\beta\gamma}{\alpha}$
$=-\alpha^2-2p+\frac{2q}{\alpha}=\alpha^2+\frac{4q}{\alpha}$

Q16cii $p=\frac{-\alpha^3-q}{\alpha}$, .: $\alpha^2=-p-\frac{q}{\alpha}$
$(\beta-\gamma)^2=\alpha^2+\frac{4q}{\alpha}$, $(\gamma-\alpha)^2=\beta^2+\frac{4q}{\beta}$, $(\alpha-\beta)^2=\gamma^2+\frac{4q}{\gamma}$
.: $(\beta-\gamma)^2=\frac{3q}{\alpha}-p$, $(\gamma-\alpha)^2=\frac{3q}{\beta}-p$, $(\alpha-\beta)^2=\frac{3q}{\gamma}-p$
$(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2=\left(\frac{3q}{\gamma}-p\right)\left(\frac{3q}{\alpha}-p\right)\left(\frac{3q}{\beta}-p\right)$
Expand and simplify
$=\frac{27q^3}{\alpha\beta\gamma}-9pq^2\left(\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}\right)+3p^2q\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right)-p^3$
$=\frac{27q^3}{-q}-9pq^2\left(\frac{0}{-q}\right)+3p^2q\left(\frac{p}{-q}\right)-p^3$
$=-(27q^2+4p^3)$

Q16ciii If $27q^2+4p^3<0$, $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2>0$
.: α, β and γ are distinct.
$p(x)$ is cubic, .: either all real solutions or one real and two complex conjugates.
Without loss of generality, consider α and β as complex conjugates and γ as real.
.: $(\beta-\gamma)^2$ and $(\gamma-\alpha)^2$ are complex conjugates, their product is real and positive.
$(\alpha-\beta)^2$ is real but negative.
.: $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2<0$, a contradiction to $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2>0$.
.: α, β and γ are distinct real roots of $p(x)=0$.

Please inform mathline@itute.com re conceptual and/or mathematical errors.