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2018 Specialist Mathematics

Year 12 Application Task

(Time allowed: 4 hours plus)

Problem Solving Task

(Time allowed: 2 hours plus)

(Assumed knowledge: Coordinate geometry, parametric equations, double angle formulas, matrices and transformations, calculus, length of a curve, area of a region, volume of solid of revolution, CAS)

Theme: Lengths and enclosed area of closed curves

Application Task (Part I and Part II, total 130 marks)
Problem Solving Task (Part I only, total 70 marks)

You are allowed: 1 bounded reference, 1 CAS, 1 scientific calculator
Working must be shown for questions worth 2 or more marks
Express answers in simplest form

Part I

Question 1
Apply the following transformations to the closed curve $x^2 + y^2 = 1$ in the order shown below.

(1) Dilation in the x-direction by a factor of $\frac{1}{2}$.

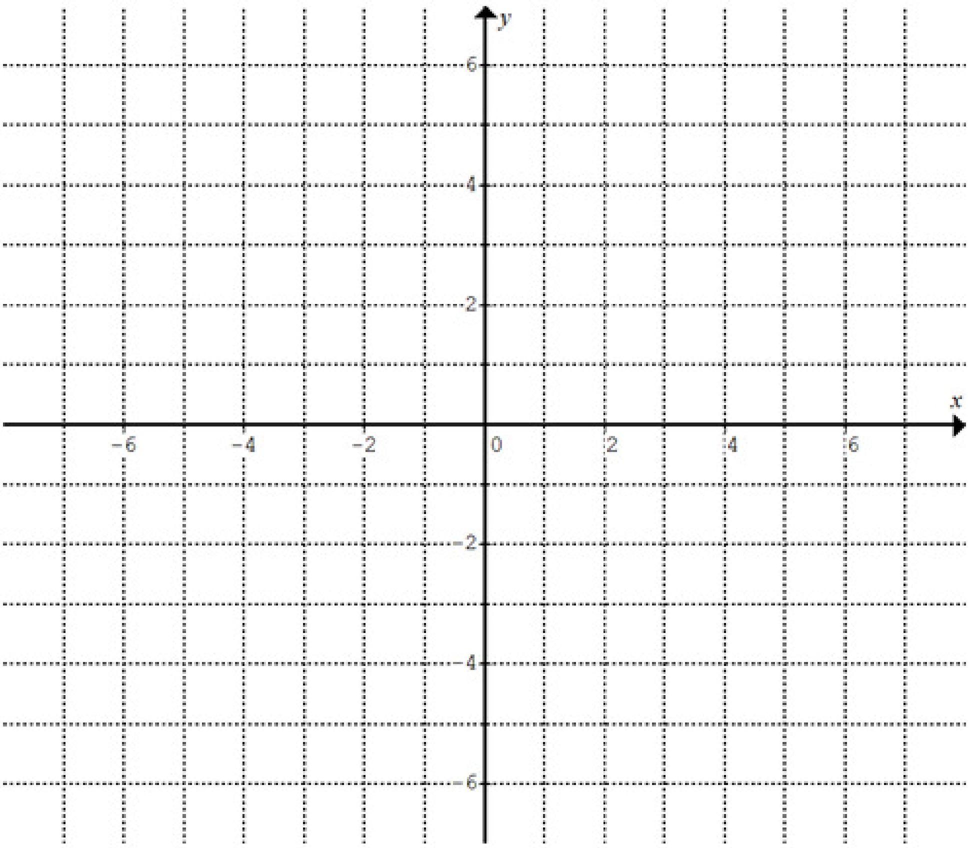
(2) Translate the resulting curve in the positive x-direction by $\frac{1}{2}$ units.

(3) Translate the resulting curve in the positive y-direction by 2 units.

(4) Dilation in the y-direction by a factor of 2.

a. Sketch the closed curve before and after the transformations listed above on the same axes shown below.

4 marks



b. Show that the equation of the closed curve after the transformations is $16\left(x-\frac{1}{2}\right)^2+(y-4)^2-4=0$.

3 marks

c. Calculate the exact area of the closed curve after the transformations. 2 marks

d. Show that the perimeter of the closed curve after the transformations is between 8 and 10 units. 3 marks

Question 2

The closed curve $16\left(x-\frac{1}{2}\right)^2+(y-4)^2-4=0$ in **Question 1** is now translated and centred at the origin.

a. Show that the equation of the closed curve centred at the origin is $4x^2+\frac{y^2}{4}=1$. 2 marks

b. Find the area of the smallest rectangle which has sides parallel to the axes and encloses the rotated closed curve completely.

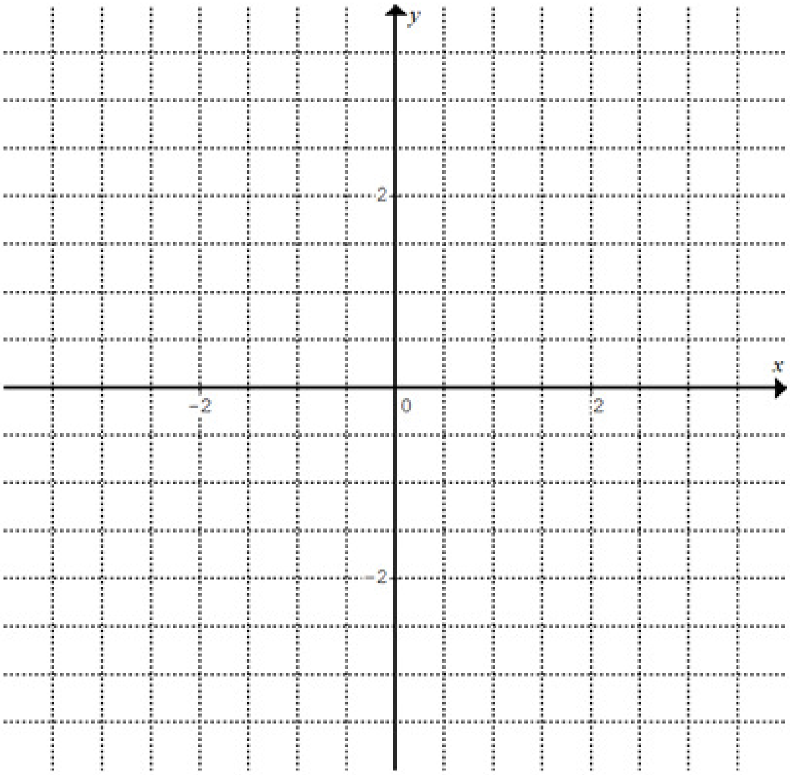
1 mark

c. The curve in part a is rotated anticlockwise about the origin by $\frac{\pi}{4}$.

The rotation matrix is $\begin{bmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}$.

Show that the equation of the closed curve after the rotation is $16(x+y)^2+(x-y)^2-8=0$. 4 marks

d. Sketch accurately the graph of the rotated closed curve. 2 marks



e. Find the exact area of the smallest rectangle which has sides parallel to the axes and encloses the rotated closed curve completely.

5 marks

f. State the possible values of the area of the smallest rectangle which has sides parallel to the axes and encloses the rotated closed curve completely.

2 marks

g. The curve in part c is rotated a further $\frac{\pi}{4}$ anticlockwise about the origin.

Show that the equation of the closed curve after the rotation is $\frac{x^2}{4}+4y^2=1$. 3 marks

h. Find the length of the closed curve in part g. Correct your answer to 4 decimal places. 4 marks

i. Find the length of the closed curve in part c. Correct your answer to 4 decimal places. 2 marks

j. Find the volume of the 3-dimensional figure formed by rotating the closed curve in part c about the *x*-axis.

5 marks

Question 3

Apply the following transformations to the closed curve $x^2 + y^2 = 1$.

(1) Dilation in the x-direction by a factor of m.
(2) Dilation in the y-direction by a factor of n.

a. In terms of m and n, find the equation of the closed curve after the transformations. 2 marks

b. In terms of m and n, state the area of the smallest rectangle which has sides parallel to the axes and encloses the rotated closed curve completely.

1 mark

c. The curve in part a is rotated anticlockwise about the origin by $\frac{\pi}{4}$.

The rotation matrix is $\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$.

In terms of m and n, find the equation of the closed curve after the rotation. 3 marks

d. In terms of m and n, find the enclosed area of the closed curve in part c. 2 marks

e. In terms of *m* and *n*, find the exact area of the smallest rectangle which has sides parallel to the axes and encloses the rotated closed curve completely.

5 marks

f. In terms of *m* and *n*, find the volume of the 3-dimensional figure formed by rotating the closed curve in part c about the line $y = x$.

5 marks

Question 4

Consider Question 3 part e and part f, given $0 < mn \leq 1$.

a. Refer to Question 3 part e.
Find the exact area of the largest smallest rectangle which has sides parallel to the axes and encloses the rotated closed curve completely.

5 marks

b. Refer to Question 3 part f.
Find the exact volume of the largest 3-dimensional figure formed by rotating the closed curve about the line $y = x$.

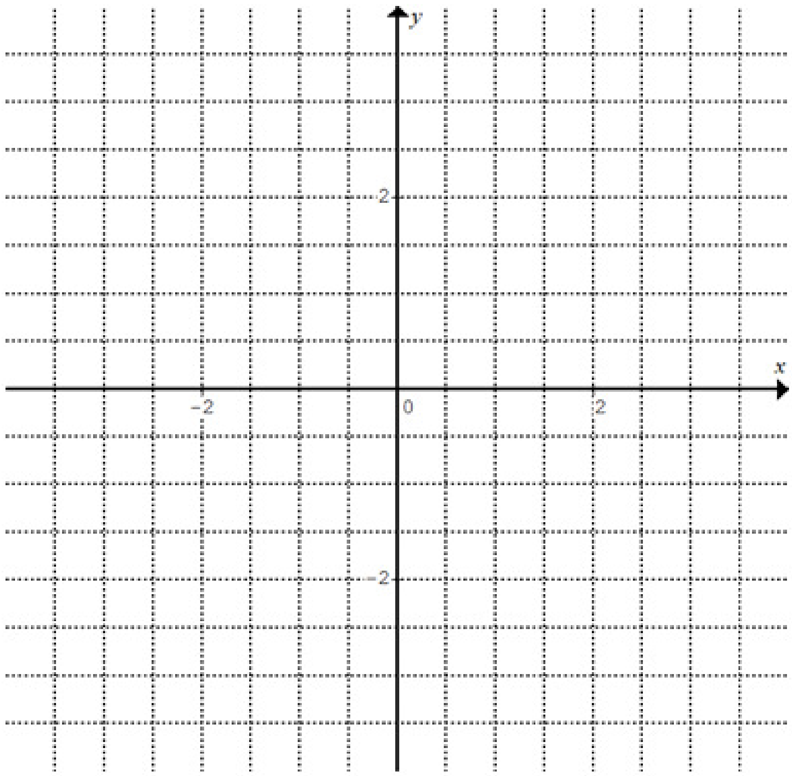
5 marks

Part II

The locus/path of a particle on the Cartesian plane is given by equations $x = a\cos(\pi t)$ and $y = b\sin(\pi t)$ where a, b and t are parameters, a, $b > 0$ and $t \geq 0$.

Question 5

a. Sketch the locus/path of a particle on the Cartesian plane (shown below) for $a = 3$ and $b = 2$. 4 marks



b. If parameter t represents time in seconds, draw an arrow head on your curve in part a to indicate the direction of motion of the particle, and mark a dot on the curve to indicate the starting point.

2 marks

c. State the shape of the locus/path, and the effects of changing parameters a and b on the locus/path.

3 marks

d. Find the exact time (in seconds) for the particle to complete five cycles. 2 marks

e. The area of the region enclosed by the locus/path of the particle is given by $A = 4 \times \int_0^3 y\,dx$.

Express the definite integral in terms of t. 4 marks

f. Evaluate your answer in part e without using CAS. 4 marks

g. Determine the total distance travelled by the particle after completing five cycles. 5 marks

Question 6

Consider the locus/path of the particle given by parametric equations $x = a\cos(\pi t)$ and $y = b\sin(\pi t)$.

Now the locus/path is rotated anticlockwise about the origin by $\frac{\pi}{4}$.

The rotation matrix is $\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$.

a. Determine the parametric equations of the locus/path of the particle after the rotation in terms of a, b and t.

3 marks

b. In terms of a and b, find the Cartesian coordinates of the particle at $t = 0$, $t = \frac{1}{2}$ and $t = \frac{3}{2}$. 3 marks

c. In terms of a and b, find the area of the region enclosed by the locus/path, using the same method as in Question 5 part e.

5 marks

d. Write a definite integral for calculating the total distance travelled by the particle after completing a cycle.

3 marks

e. In terms of a and b, find a formula for calculating the total distance travelled by the particle after completing a cycle.

3 marks

f. Use the formula in part e to answer Q5 part f. 2 marks

Question 7

Consider the locus/path of the particle given by parametric equations $x = a\cos(\pi t)$ and $y = b\sin(\pi t)$.
Now the locus/path is rotated anticlockwise about the origin by angle θ.

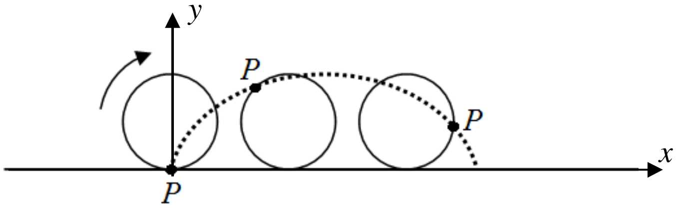
a. Determine the parametric equations of the locus/path of the particle after the rotation in terms of a, b, θ and t.

4 marks

b. Given $a = 3$, $b = 2$ and $\theta = \frac{\pi}{4}$, by substitution into the equations in part a, and evaluation by hand, determine the Cartesian coordinates of the particle at $t = 0$, $t = \frac{1}{2}$ and $t = \frac{3}{2}$. 3 marks

Question 8

The dotted locus/path traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid**.



If the radius of the circle is 1 unit, and point P is at the origin $(0, 0)$ when $t = 0$, the parametric equations for the cycloid are:

$$x = \pi t - \sin(\pi t)$$
$$y = 1 - \cos(\pi t)$$

a. Find the time taken by the circle to complete one complete turn. 1 mark

b. Find the coordinates of P when it touches the x-axis again. 1 mark

c. Find the length of the cycloid after 5 complete turns of the circle. 4 marks

d. Find the area of the region enclosed by the cycloid and the x-axis after 5 complete turns of the circle. 4 marks

End of task