2018 VCAA Mathematical Methods Exam 2 Solutions

Use CAS to save time

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
C	A	D	C	A	D	B	B	C	C

11	12	13	14	15	16	17	18	19	20
C	E	E	B	E	B	C	E	C	A

Q1 Period $=\frac{2\pi}{\frac{2\pi}{3}}=3$ C

Q2 A

Q3 D

Q4 C

Q5 $f'(-2)=2(-2)-\frac{p}{(-2)^2}=0$, $p=-16$ A

Q6 $g(x+2)=3x+1$, $g((x-2)+2)=3(x-2)+1$, $g(x)=3x-5$

$f(x)=2x$, $f(g(x))=2g(x)=6x-10$ D

Q7 $x=k\log_2(y)$, $y=f^{-1}(x)=2^{\frac{x}{k}}$, $f^{-1}(1)=2^{\frac{1}{k}}=8$, $k=\frac{1}{3}$ B

Q8 $\int_5^{12} g(x)dx=6$, $\int_1^{12} g(x)dx=\int_1^5 g(x)dx+\int_5^{12} g(x)dx$

$5=\int_1^5 g(x)dx+6$, $\int_1^5 g(x)dx=-1$ B

Q9 $\frac{dy}{dx}=\frac{1}{x}=2$, .: $x=\frac{1}{2}$, $y=0$

Tangent: $y-0=2\left(x-\frac{1}{2}\right)$, $y=2x-1$ C

Q10 $x+f(x)=2x$, $f(x)=x$ C

Q11 $y=\tan\left(\frac{x}{2}\right)$, $a=\frac{1}{2}$, x-intercept at $x=2\pi$ C

Q12 $\mu=\frac{17}{10}$, $\Pr\left(X<\frac{17}{10}\right)=\Pr(X=0)+\Pr(X=1)=\frac{7}{10}$ E

Q13 $\Pr(+1)=\Pr(rw)+\Pr(wr)=\frac{2}{6}\times\frac{2}{5}+\frac{4}{6}\times\frac{3}{5}=\frac{8}{15}$ E

Q14 $\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A)\Pr(B)$

$0.52=\Pr(A)+2\Pr(A)-2\Pr(A)\Pr(A)$, $0.52=3\Pr(A)-2(\Pr(A))^2$

$2(\Pr(A))^2-3\Pr(A)+0.52=0$, $\Pr(A)=0.2$ B

Q15 $\int_0^m \frac{1}{12}(8x-x^3)dx=0.5$, $\left[\frac{1}{12}\left(4x^2-\frac{x^4}{4}\right)\right]_0^m=0.5$, $4m^2-\frac{m^4}{4}=6$

$m^4-16m^2+24=0$ E

Q16 Approx area $=\frac{\pi}{6}(4+2+1)=\frac{7\pi}{6}$

Exact area $\int_0^{\frac{\pi}{2}}(2\cos(2x)+3)dx=[\sin(2x)+3x]_0^{\frac{\pi}{2}}=\frac{3\pi}{2}$

Fraction $=\frac{7\pi}{6}\div\frac{3\pi}{2}=\frac{7}{9}$ B

Q17 TP: $x=-\frac{-2b}{2}=b$, $y=1-b^2$

$D=\sqrt{b^2+(1-b^2)^2}=\sqrt{b^4-b^2+1}$, $\frac{dD}{db}=0$, $b=\pm\frac{1}{\sqrt{2}}$ C

Q18 $f(x)>g(x)$ for $x\in(0,1)$ and $g(x)>f(x)$ for $x\in(1,\infty)$

.: $x^{\frac{p}{q}}>x^{\frac{m}{n}}$ for $x\in(0,1)$ and $x^{\frac{m}{n}}>x^{\frac{p}{q}}$ for $x\in(1,\infty)$

$\frac{p}{q}<\frac{m}{n}$ for $x\in(0,1)$ and $\frac{m}{n}>\frac{p}{q}$ for $x\in(1,\infty)$

.: $pn<qm$ for $x\in(0,1)$ or $x\in(1,\infty)$

If $p=m$, $q>n$ for $x\in(0,1)$ or $x\in(1,\infty)$

If $q=n$, $m>p$ for $x\in(0,1)$ or $x\in(1,\infty)$

.: A, B and C are true statements

The graphs of $f(x)=x^{\frac{p}{q}}$ and $g(x)=x^{\frac{m}{n}}$ intersects at $x=0$ and $x=1$.

It is true that $f'(c)=g'(c)$ for some $c\in(0,1)$.

After $x=1$ the two graphs diverge, .: $f'(d)\neq g'(d)$ for some $d\in(1,\infty)$ E

Q19 C

Q20 $\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}-2&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$, $x'=-2x$ and $y'=\frac{1}{2}y$

.: $y=2y'$ and $x=-\frac{1}{2}x'$, $y=f(x)\to 2y'=f\left(-\frac{1}{2}x'\right)$

i.e. $y=\frac{1}{2}f\left(-\frac{1}{2}x\right)=g(x)$, .: $g'(x)=-\frac{1}{4}f'\left(-\frac{1}{2}x\right)$

.: $g'(0)=-\frac{1}{4}f'(0)=-1$ A

SECTION B

Q1a Let $f'(x)=12x^3+12x^2-24x=12x(x+2)(x-1)=0$

$x=-2,0,1$, $y=f(-2)=-32$, .: $M(-2,-32)$

Q1b $b>32$

Q1c $x=-\frac{1}{3}$, $y=f\left(-\frac{1}{3}\right)=\frac{3}{81}-\frac{4}{27}-\frac{4}{3}=-\frac{13}{9}$, $f'\left(-\frac{1}{3}\right)=\frac{80}{9}$

Tangent: $y+\frac{13}{9}=\frac{80}{9}\left(x+\frac{1}{3}\right)$, $y=\frac{80}{9}x+\frac{41}{27}$

Q1d $3x^4+4x^3-12x^2=\frac{80}{9}x+\frac{41}{27}$

$81x^4+108x^3-324x^2-240x-41=0$

$(3x+1)^2(9x^2+6x-41)=0$

$x=\frac{-6\pm\sqrt{36+1476}}{18}=\frac{-1\pm\sqrt{42}}{3}$

Q1e Area $=\int_{\frac{-1-\sqrt{42}}{3}}^{\frac{-1+\sqrt{42}}{3}}\left(\frac{80}{9}x+\frac{41}{27}-3x^4-4x^3+12x^2\right)dx$

$=\left[\frac{40}{9}x^2+\frac{41}{27}x-\frac{3}{5}x^5-x^4+4x^3\right]_{\frac{-1-\sqrt{42}}{3}}^{\frac{-1+\sqrt{42}}{3}}$

$=\frac{784\sqrt{42}}{135}$

Q1f $3x^4+4x^3-12x^2=3x^4+4x^3+6(a-2)x^2-12ax+a^2$

.: $a^2=0$, $-12a=0$ and $6a=0$, .: $a=0$ for all x

Q1g $p'(x)=12x^3+12x^2+12(a-2)x-12a=0$

$x^3+x^2+(a-2)x-a=0$

$x=1$ or $-1\pm\sqrt{1-a}$

Q1hi $1-a<0$, $a>1$

Q1hii When $a=2$, $p(x)=3x^4+4x^3-24x+4$, min $p=-13$

Q1hiii $x=1$ and $a>1$

Let $p(1)=6(a-2)-12a+a^2+7>0$, .: $a>3+\sqrt{14}$

Q2a $b(t)=\frac{4500}{7}\left(e^{-\frac{t}{5}}-e^{-\frac{9t}{10}}\right)$, $t\geq 0$

Let $b'(t)=\frac{4500}{7}\left(-\frac{1}{5}e^{-\frac{t}{5}}+\frac{9}{10}e^{-\frac{9t}{10}}\right)=0$

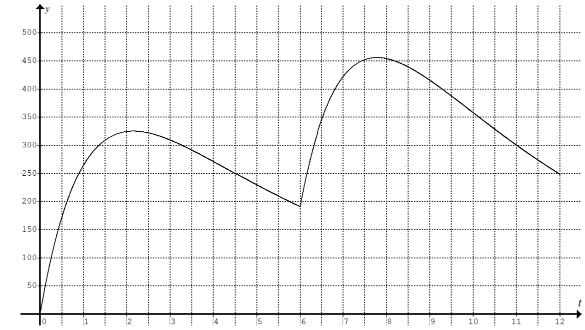
$\frac{9}{10}e^{-\frac{9t}{10}}=\frac{1}{5}e^{-\frac{t}{5}}$, $e^{\frac{7t}{10}}=\frac{9}{2}$, $t=\frac{10}{7}\log_e\left(\frac{9}{2}\right)$

Q2b Av rate $=\frac{4500}{7}\left(\frac{e^{-\frac{6}{5}}-e^{-\frac{54}{10}}-e^{-\frac{2}{5}}+e^{-\frac{18}{10}}}{6-2}\right)\approx -33.5$ mg per hour

Q2c Av amount $=\frac{\int_0^6 \frac{4500}{7}\left(e^{-\frac{t}{5}}-e^{-\frac{9t}{10}}\right)dt}{6}\approx 256$ mg

Q2di After the first 6 hours,

$b(t)=\frac{4500}{7}\left(e^{-\frac{t}{5}}-e^{-\frac{9t}{10}}\right)+\frac{4500}{7}\left(e^{-\frac{t-6}{5}}-e^{-\frac{9(t-6)}{10}}\right)$ for $t\geq 6$



Q2dii Max amount ≈ 455.82 mg when $t=7.78$ h

Q3a $a=75$

Q3b $h_3(x)$ is the translation of $h_2(x)$ by 35 units (m) in the positive x-direction.

Q3c Area of the shaded region

$=5\times 110-6\times\int_5^{20}5\sin\frac{(x-5)\pi}{30}dx\approx 264\text{ m}^2$

Q3d Gradient $=\tan\left(\frac{\pi}{90}\right)\approx 0.035$

Q3e $h_5(x)=5\sin\frac{(x-40)\pi}{30}$,

Let $h_5'(x)=\frac{\pi}{6}\cos\frac{(x-40)\pi}{30}=0.035$, $x=54.3626$, $y\approx 4.99$

.: $P(54.36, 4.99)$

Q3f Tangent at *P*: $\frac{y-4.99}{x-54.36}=0.035$, y-intercept is $y\approx 3.0874$

$PQ\approx(5-3.0874)\cos\left(\frac{\pi}{90}\right)\approx 1.91$ m

Q4a $\Pr(60 < X < 90) \approx 0.838$

Q4bi $\Pr(H \mid S) = \dfrac{\Pr(H \cap S)}{\Pr(S)} = \dfrac{0.09}{0.29} \approx 0.310$

Q4bii $\Pr(H) = 0.1587$

$\therefore$ $\Pr(H \mid S) \neq \Pr(H)$, $\therefore$ H and S are not independent.

Q4ci $\Pr(\text{exactly } 1) = \binom{16}{1} 0.1587 \times (1 - 0.1587)^{15} \approx 0.190$

Q4cii

$\Pr(\hat{P} > 0.10) = \Pr(X > 1.6) = \Pr(X \geq 2) = 1 - \Pr(X \leq 1) \approx 0.747$

Q4ciii

$\Pr\left(\hat{P}_n > \dfrac{1}{n}\right) = \Pr\left(\dfrac{X}{n} > \dfrac{1}{n}\right) = \Pr(X > 1) = 1 - \Pr(X \leq 1) > 0.99$

$\Pr(X \leq 1) < 0.01$,

$\binom{n}{0} 0.1587^0 (1 - 0.1587)^n + \binom{n}{1} 0.1587^1 (1 - 0.1587)^{n-1} < 0.01$

$0.8413^n + n(0.1587)(0.8413)^{n-1} < 0.01$

$\therefore$ $n > 38.93$, least $n = 39$

Q4di $2\hat{p} = 0.102 + 0.145$, $\hat{p} = 0.1235$

Q4dii 95% chance that $0.102 < P_{\text{Statsville}} < 0.145$

$P_{\text{Mathsland}} = 0.1587$ which is outside the interval $(0.102, 0.145)$

Q4e $E(T) = \displaystyle\int_0^n \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} t \, dt \approx 44.6$ min as $n \to \infty$

Q4f $\Pr(T < 15) = \displaystyle\int_0^{15} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} dt \approx 0.0266$

Q4g $\Pr(\text{non Yr12}) = \dfrac{6}{7}$, $\Pr(\text{elite} \mid \text{Yr12}) = 0.05$

	elite	not elite	
Yr12	$\frac{0.05}{7}$	$\frac{0.95}{7}$	$\frac{1}{7}$
Non-yr12	0.019457	0.83764	$\frac{6}{7}$
	0.0266	0.9737	1

$\Pr(\text{elite} \mid \text{non Yr12}) = \dfrac{\Pr(\text{elite} \cap \text{non Yr12})}{\Pr(\text{non Yr12})} \approx \dfrac{0.019457}{\frac{6}{7}} \approx 0.0227$

Q5a Let $f'(x) = \dfrac{81}{4a^4}\left(2ax - 3x^2\right) = 0$, $x = 0$ or $\dfrac{2a}{3}$

Local max at $x = \dfrac{2a}{3}$, $y = \dfrac{3}{a}$

Local max is $\left(\dfrac{2a}{3}, \dfrac{3}{a}\right)$

Q5b $\dfrac{81x^2(a - x)}{4a^4} = \dfrac{9x}{2a^2}$, $81x^2(a - x) - 18a^2 x = 0$

$9x\left(9x(a - x) - 2a^2\right) = 0$, $-9x\left(9x^2 - 9ax + 2a^2\right) = 0$

$\therefore$ $x = 0$, $\dfrac{a}{3}$, $\dfrac{2a}{3}$

Q5c The two enclosed regions have the same area.

Total area $= 2 \times \displaystyle\int_0^{\frac{a}{3}} \left(\frac{9x}{2a^2} - \frac{81x^2(a - x)}{4a^4}\right) dx = \frac{1}{8}$

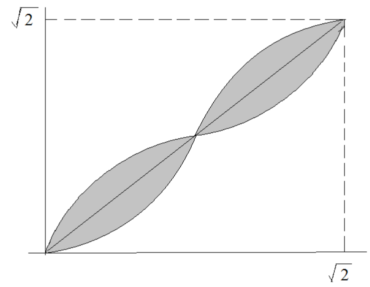
Q5d $\dfrac{2a}{3} \times g\left(\dfrac{2a}{3}\right) = \dfrac{1}{2}$

Q5e

Area $= \dfrac{1}{2} \times \dfrac{3}{a} \times \dfrac{2a}{3} = 1$

Q5f $\dfrac{2a}{3} = \dfrac{3}{a}$, $a^2 = \dfrac{9}{2}$, $a = \dfrac{3\sqrt{2}}{2}$

Q5g End point $\left(\sqrt{2}, \sqrt{2}\right)$



From part c, area $= 2 \times \dfrac{1}{8} = \dfrac{1}{4}$

Please inform mathline@itute.com re conceptual and/or mathematical errors