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2019 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

Problem Solving Task

Time allowed: 2.6 hours plus

2019 Mathematical Methods
Problem Solving Task (Parts I and II)
Application Task (Parts I, II and III)

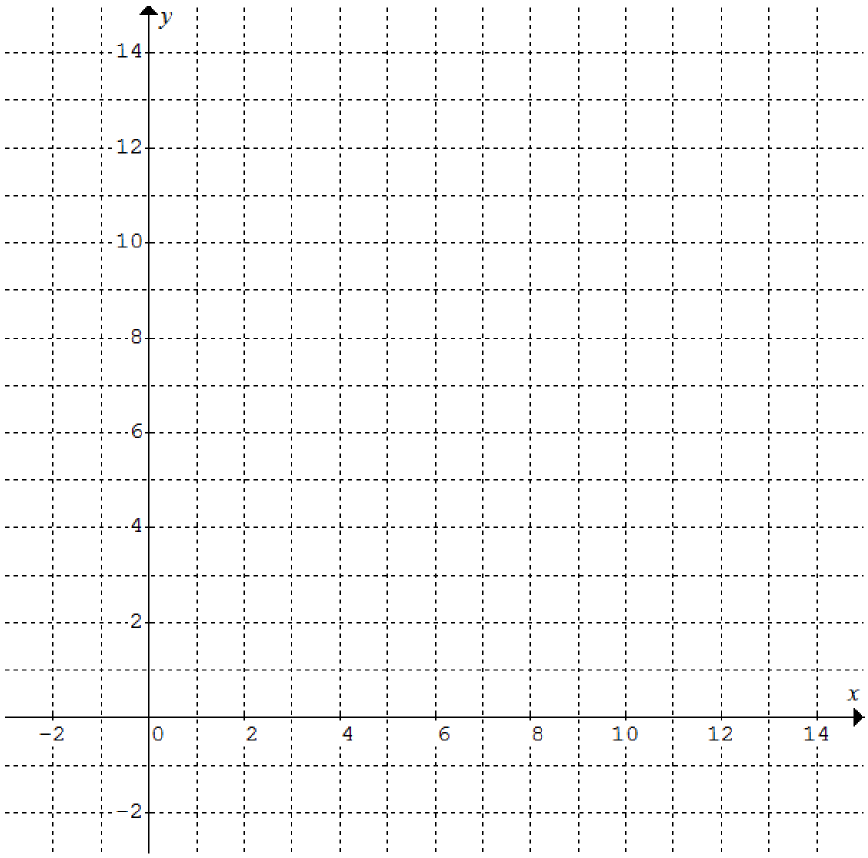
Theme: Roots of equation involving linear and exponential or logarithmic functions

Assumed knowledge: Linear, exponential and logarithmic functions, roots of equation, transformations and inverse functions, gradient and area of region between curves

Part I (40 marks)

a. Sketch neatly and accurately on the same Cartesian plane the graphs of $y = e^x$ and $y = mx$ for $m = 1, 2, 3$ and 4.

4 marks



b. The graphs of $y = mx$ and $y = e^x$ intersect at exactly one point when $m = e$.
Use calculus to show that $m = e$.

3 marks

c. Determine the exact coordinates of the point of intersection in part **b**. 1 mark

d. For each value of $m = 3, 4$ and 5, find the roots of $mx = e^x$, correct to 4 decimal places. Check that in each case the sum of the roots is greater than 2. 4 marks

e. For $m = 3$, find the area of the region bounded by the graphs of $y = mx$ and $y = e^x$.

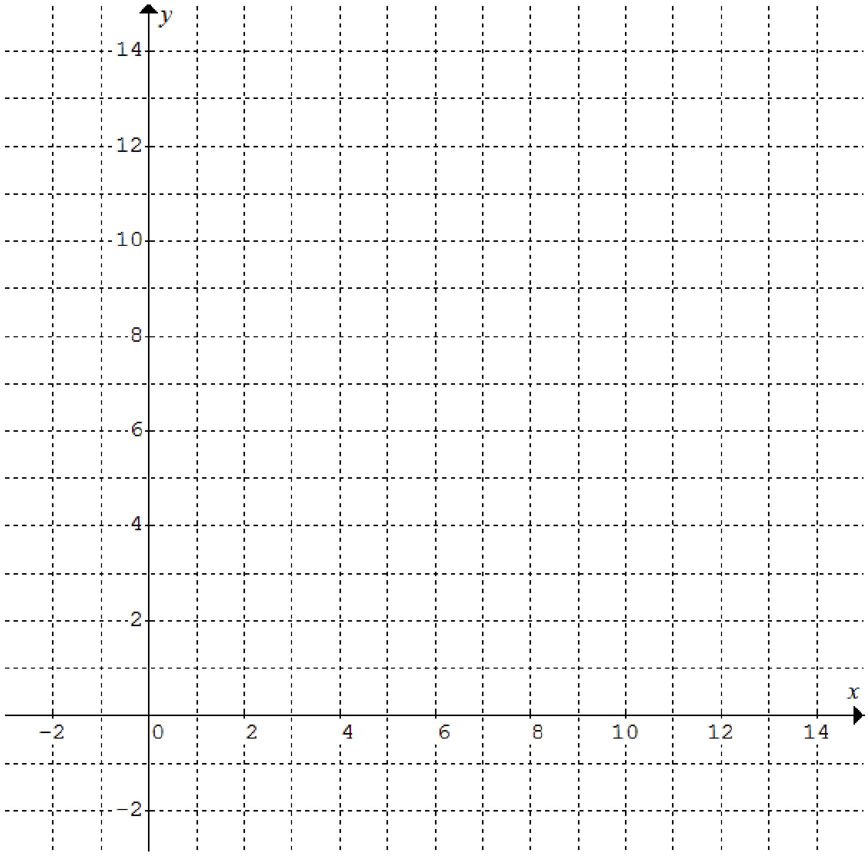
2 marks

f. The graphs of $y = mx$ and $y = e^{x-\alpha}$ intersect at exactly one point for $\alpha > 0$ and $m > 0$. Use the result in part **b** to find the values of α and m, and the coordinates of the point of intersection.

4 marks

g. Sketch neatly and accurately on the same Cartesian plane the graphs of $y = \log_e x$ and $y = nx$ for $n = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ and 1.

4 marks



h. The graphs of $y = \log_e x$ and $y = nx$ intersect at exactly one point when $n = e^{-1}$.
Use calculus to show that $n = e^{-1}$.

3 marks

i. Determine the exact coordinates of the point of intersection in part **h**.

1 mark

j. For each value of $n=\frac{1}{5},\frac{1}{4}$ and $\frac{1}{3}$, find the roots of $\log_e x = nx$, correct to 4 decimal places.

Check that in each case the sum of the roots is greater than $\frac{2}{n}$. 4 marks

k. For $n=\frac{1}{3}$, find the area of the region bounded by the graphs of $y=\log_e x$ and $y=nx$.

2 marks

l. The graphs of $y=\log_e x+\beta$ and $y=nx$ intersect at exactly one point for $\beta>0$ and $n>0$.

Calculate the values of β and n, and the coordinates of the point of intersection.

5 marks

m. Use the results in part **f** and part **l** to find the root(s) of $e^{x-1}=\log_e x+1$. 3 marks

Part II (40 marks)

Equation of the form $mx = e^x$, where $m > e$, has two real roots as discussed in **Part I**.

In general, the exact roots of equation cannot be found algebraically unless $m = \frac{e^a}{a}$, and the exact root is $x = a$, $a \in R$.

a. Equation $mx = e^x$, where $m > e$, has $x = 0.75$ as its exact root.
Find the exact value of m, and the other root correct to 4 decimal places.

3 marks

b. Find the exact real root of equation $2x\left(\sqrt{e}\right)^3 = 3e^x$, and the other root correct to 4 decimal places.

4 marks

c. Make up an equation of the form $mx = e^x$ which has an exact real root.
Find the roots of your equation.

3 marks

In **Part I** you have checked that the sum of the roots of equation $mx = e^x$ for $m > e$ is greater than 2.

d. Confirm that it is true for the equations in **Part II a**, **b** and **c**. 2 marks

e. In terms of gradients explain that $mx > e^x$ for $x \in (x_1, x_2)$ where x_1 and x_2 are the roots of $mx = e^x$.

3 marks

f. Let x_1 and x_2 be the roots of $mx = e^x$.
In terms of x_1 and x_2 find the area A of the region bounded by the graphs of $y = mx$ and $y = e^x$.

3 marks

g. Express A in part **f** as a polynomial function of x_1 and x_2. 2 marks

h. Show that $A = \frac{m}{2}(x_2 - x_1)(x_2 + x_1 - 2)$. 3 marks

i. Hence show that the sum of the roots of $mx = e^x$ is greater than 2. 2 marks

j. Let x_1 and x_2 be the roots of $mx = he^x$ where $m, h \in R$.
Determine the values of $x_1 + x_2$. 4 marks

k. Determine the relation between m and h in $mx = he^x$. 2 marks

l. Let x_1 and x_2 be the roots of $mx - m = he^x$ where $m, h \in R$.
Determine the values of $x_1 + x_2$. 6 marks

m. Determine the relation between m and h in $mx - m = he^x$. 3 marks

Part III (35 marks)

Equation of the form $\log_e x = nx$, where $n < e^{-1}$, has two real roots as shown in **Part I**.

In general, the exact roots of equation cannot be found algebraically unless $n = \frac{\log_e b}{b}$, and the exact root is $x = b$ where $b > 1$.

a. Equation $\log_e x = nx$, where $n < e^{-1}$, has $x = 2$ as its exact root and $n = \log_e w$.
Find the exact value of w, and the other root correct to 4 decimal places.

3 marks

b. Equation $e^2 \log_e x = 2x$ has an exact real root.
Find this exact root of the equation and the other root correct to 4 decimal places.

5 marks

c. Make up an equation of the form $\log_e x = nx$ which has an exact real root.
Find the roots of your equation.

3 marks

In **Part I** you have checked that the sum of the roots of equation $\log_e x = nx$ for $n < e^{-1}$ is greater than $\frac{2}{n}$.

d. Confirm that it is true for the equations in **Part III a**, **b** and **c**. 2 marks

e. In terms of gradients explain that $\log_e x > nx$ for $x \in (x_1, x_2)$ where x_1 and x_2 are the roots of $\log_e x = nx$.

2 marks

f. Let x_1 and x_2 be the roots of $\log_e x = nx$.
Show that the area A of the region bounded by $y = \log_e x$ and $y = nx$ is given by
$A = \frac{n}{2}\left(x_2{}^2 - x_1{}^2\right) - \left(e^{nx_2} - e^{nx_1}\right)$. 7 marks

g. Express A in part **f** as a polynomial function of x_1 and x_2. 3 marks

h. Show that $A=(x_2-x_1)\left(\frac{n}{2}(x_2+x_1)-1\right)$. 3 marks

i. Hence show that the sum of the roots of $\log_e x = nx$ is greater than $\frac{2}{n}$. 2 marks

j. Let x_1 and x_2 be the roots of $k\log_e x = nx$ where $n, k \in R$.
Determine the values of $x_1 + x_2$ in terms of n and k. 3 marks

k. Determine the relation between n and k in $k\log_e x = nx$. 2 marks

End of Task

