2019 NSW ESA Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	D	C	B	A	B	C	A	D	A

Q1 $4-x>0$, $x<4$ **A**

Q2 $(2x)^2=x(9+x)$, $4x^2=x(9+x)$ **D**

Q3 $\frac{d}{dx}\tan^{-1}\frac{x}{2}=\frac{2}{2^2+x^2}=\frac{2}{4+x^2}$ **C**

Q4 $y=\frac{x^2}{x^2-1}=1+\frac{1}{x^2-1}$ **B**

Q5 Period $=2$, at $t=0$, $x=2$ and $\frac{dx}{dt}=0$ **A**

Q6 When $\frac{\pi}{2}<x<\pi$, $\cos x=-\sqrt{1-\sin^2 x}=-\frac{\sqrt{15}}{4}$

$\sin 2x=2\sin x\cos x=-\frac{\sqrt{15}}{8}$ **B**

Q7 Let β be the third zero. $(-1)\alpha\beta=-\frac{q}{q}$, .: $\beta=\frac{1}{\alpha}$ **C**

Q8 Let X represents LLL

Number of permutations of $XAAEPR$ is $\frac{6!}{2!}$ **A**

Q9 $y=\cos^{-1}(-\sin x)=\cos^{-1}(\sin(-x))=\cos^{-1}\left(\cos\left(\frac{\pi}{2}-(-x)\right)\right)$

$=\cos^{-1}\left(\cos\left(\frac{\pi}{2}+x\right)\right)=x+\frac{\pi}{2}$ **D**

Q10 $f(x)=-\sqrt{1+\sqrt{1+x}}$ is defined for $1+x\geq 0$, .: $x\geq -1$ and $y\leq -1$.: for $f^{-1}(x)$, $x\leq -1$ and $y\geq -1$

.: one point in common which is $(-1,-1)$. **A**

Section II

Q11a $m_1=\frac{1}{2}$, $m_2=3$, $\tan\theta=\frac{3-\frac{1}{2}}{1+3\left(\frac{1}{2}\right)}=1$.: $\theta=\frac{\pi}{4}$

Q11b Consider the graph of $y=\frac{x}{x+1}=1-\frac{1}{x+1}$.

Asymptotes are $y=1$ and $x=-1$.

$y=\frac{x}{x+1}$ and $y=2$ intersect at $x=-2$

.: $y=\frac{x}{x+1}<2$ for $x<-2$ or $x>-1$.

Q11c $\angle BCA$ is a right angle (subtended by a diameter).

.: $\angle BAC+\angle ABC=\angle BCD+\angle CBD=90°$

Since $\angle BAC=\angle BCD$ (angles in alternate segments)

.: $\angle ABC=\angle CBD$

Q11d

$Q(x)=\frac{x^3+2x^2-3x-10}{x-2}=\frac{(x-2)(x^2+4x+5)}{x-2}=x^2+4x+5$

Q11e $\int 2\sin^2 4x\,dx=\int 1-\cos 8x\,dx=x-\frac{1}{8}\sin 8x+c$

Q11fi $\Pr(0)=\left(1-\frac{5}{100}\right)^8=\left(\frac{19}{20}\right)^8$ or ≈ 0.66

Q11fii $\Pr(\text{at least }2)=1-\Pr(0)-\Pr(1)=1-\left(\frac{19}{20}\right)^8-8\left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^7$

≈ 0.057

Q12a $A=\frac{9}{B}$ When $A=12$, $B=\frac{3}{4}$

$\frac{dA}{dt}=-\frac{9}{B^2}\times\frac{dB}{dt}=-\frac{9}{\left(\frac{3}{4}\right)^2}\times 0.2=-3.2$ m s^{-1}

Q12bi $x=R\cos(3t-\alpha)=R\cos 3t\cos\alpha+R\sin 3t\sin\alpha$ where

$R\cos\alpha=\sqrt{2}$ and $R\sin\alpha=\sqrt{6}$, .: $R=2\sqrt{2}$ and $\alpha=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$

.: $x=2\sqrt{2}\cos\left(3t-\frac{\pi}{3}\right)$

Q12bii $x=\pm 2\sqrt{2}$

Q12biii $\dot{x}=-6\sqrt{2}\sin\left(3t-\frac{\pi}{3}\right)$, maximum speed $=6\sqrt{2}$

Let $\sin\left(3t-\frac{\pi}{3}\right)=\pm\frac{1}{2}$, $3t-\frac{\pi}{3}=-\frac{\pi}{6}$ or $\frac{\pi}{6}$, .: $t=\frac{\pi}{18}$ or $\frac{\pi}{6}$

$t=\frac{\pi}{18}$ is the first time.

Q12c At $P(2ap,ap^2)$, $\frac{dy}{dx}=\frac{x}{2a}=p$, tangent: $y=p(x-2ap)+ap^2$

and y-intercept $R(0,y)$ where $y=-ap^2$. .: $SR=a+ap^2$

$SP=PM=ap^2+a=SR$

.: ΔSRP is isosceles and $\angle SPR=\angle SRP$

Q12di $T=3+Ae^{kt}$, $\frac{dT}{dt}=kAe^{kt}$, $\frac{dT}{dt}=k(T-3)$

Q12dii $t=15$, $T=28$, $15k=\ln\left(\frac{28-3}{27}\right)$, $k=\frac{1}{15}\ln\left(\frac{25}{27}\right)$

$t=60$, $T=3+27e^{60k}$, $\frac{dT}{dt}=k(T-3)=\frac{1}{15}\ln\left(\frac{25}{27}\right)\times 27\left(\frac{25}{27}\right)^4$

$\frac{dT}{dt}=\frac{5}{3}\left(\frac{25}{27}\right)^3\ln\left(\frac{25}{27}\right)$

Q13a $u=\cos^2 x$, $\frac{du}{dx}=-2\cos x\sin x=-\sin 2x$

$$\int_0^{\frac{\pi}{4}}\frac{-1}{4+u}\frac{du}{dx}dx=\int_1^{\frac{1}{2}}\frac{-1}{4+u}du=\int_{\frac{1}{2}}^{1}\frac{1}{4+u}du=\left[\ln(4+u)\right]_{\frac{1}{2}}^{1}=\ln\frac{10}{9}$$

Q13b The 2 consecutive terms: ${}^{20}C_k(5x)^k 2^{20-k}$, ${}^{20}C_{k+1}(5x)^{k+1}2^{19-k}$

Equate the coefficients: ${}^{20}C_k(5)^k 2^{20-k} = {}^{20}C_{k+1}(5)^{k+1}2^{19-k}$

$2\times{}^{20}C_k=5\times{}^{20}C_{k+1}$, .: $\frac{2\times 20!}{k!(20-k)!}=\frac{5\times 20!}{(k+1)!(19-k)!}$

$\frac{2}{(20-k)}=\frac{5}{(k+1)}$, $2k+2=100-5k$, $7k=98$, $k=14$

Q13ci $a=-2e^{-x}$, $\frac{d\frac{1}{2}v^2}{dx}=-2e^{-x}$, $\frac{1}{2}v^2=-2\int e^{-x}dx=2e^{-x}+c$

$t=0$, $x=0$, $v=2$, $\frac{1}{2}v^2=2e^{-x}$, $v=\sqrt{4e^{-x}}=2e^{-\frac{x}{2}}$

Q13cii $\frac{dx}{dt}=2e^{-\frac{x}{2}}$, $\frac{dt}{dx}=\frac{1}{2}e^{\frac{x}{2}}$, $t=\frac{1}{2}\int e^{\frac{x}{2}}dx=e^{\frac{x}{2}}+c$

$t=0$, $x=0$, $v=2$, $t=e^{\frac{x}{2}}-1$, $x=2\ln(t+1)$

Q13di $y=-x$, $18\sqrt{3}t-5t^2=-18t$, $18\left(\sqrt{3}+1\right)t-5t^2=0$

$\left(18\left(\sqrt{3}+1\right)-5t\right)t=0$, $t=\frac{18\left(\sqrt{3}+1\right)}{5}$, $x=18t=\frac{18^2\left(\sqrt{3}+1\right)}{5}$

$OA\cos 45°=\frac{324\left(\sqrt{3}+1\right)}{5}$, $OA=\frac{324\sqrt{2}\left(\sqrt{3}+1\right)}{5}$

$A\left(\frac{324\left(\sqrt{3}+1\right)}{5},-\frac{324\left(\sqrt{3}+1\right)}{5}\right)$

Q13dii

At A, $v_x=18$, $v_y=18\sqrt{3}-10t=18\sqrt{3}-36\sqrt{3}-36=-18\left(2+\sqrt{3}\right)$

Let θ be the angle with the horizontal.

$\theta=\tan^{-1}\left(\frac{-18\left(2+\sqrt{3}\right)}{18}\right)=-75°$, i.e. $75°$ below the horizontal or $30°$ with $y=-x$ at point A.

Q14a For $n=1$, $LS=1(1!)=1$, $RS=2!-1=1$, .: true

Assume it is true for $n=k$,

.: $1(1!)+2(2!)+3(3!)+...+k(k!)=(k+1)!-1$

Consider $n=k+1$,

$LS=1(1!)+2(2!)+3(3!)+...+k(k!)+(k+1)(k+1)!$

$=(k+1)!-1+(k+1)(k+1)!$

$=(k+1)!(1+(k+1))-1$

$=(k+2)(k+1)!-1$

$=(k+2)!-1$

.: it is true for $n=k+1$.

Hence true for all $n\ge 1$.

Q14bi $y=x^2\ge 0$ and strictly increasing for $x>0$

$y=\frac{1}{x-k}>0$ for $x>k$ and strictly decreasing from ∞

.: the two graphs meet at only one point at $x>k$.

.: $x^2=\frac{1}{x-k}$, $x^3-kx^2-1=0$ has exactly one real zero.

Q14bii $x_2\approx x_1-\frac{f(x_1)}{f'(x_1)}=k-\frac{-1}{3k^2-2k^2}=k+\frac{1}{k^2}$

Q14biii $f(\alpha)=0$, $f(x_1)=f(k)=-1<0$

$f(x_2)=f\left(k+\frac{1}{k^2}\right)=\left(k+\frac{1}{k^2}\right)^3-k\left(k+\frac{1}{k^2}\right)^2-1$

$=\left(k+\frac{1}{k^2}\right)^2\left(k+\frac{1}{k^2}-k\right)-1=\left(\frac{k^3+1}{k^2}\right)^2\frac{1}{k^2}-1$

$=\left(\frac{k^3+1}{k^3}\right)^2-1>0$.: $x_1<\alpha<x_2$

Q14ci $\sin x=\sin(x-\alpha)+k$, $\frac{d}{dx}\sin x=\frac{d}{dx}(\sin(x-\alpha)+k)$

$\cos x=\cos(x-\alpha)$, .: $\cos x_0=\cos(x_0-\alpha)$

Q14cii $\sin(x_0-\alpha)+k=\sin x_0$, $\cos(x_0-\alpha)=\cos x_0$

Square both sides and add: $2k\sin(x_0-\alpha)+k^2=0$

.: $k(2\sin(x_0-\alpha)+k)=0$ and $k>0$

.: $k=-2\sin(x_0-\alpha)$, .: $\sin x_0=-\sin(x_0-\alpha)$

Q14ciii .: $\sin x_0=\sin(\alpha-x_0)$, .: $x_0=\alpha-x_0$ since $0<x_0<\frac{\pi}{2}$ and $0<\alpha<\pi$.: $x_0=\frac{\alpha}{2}$ and $k=-2\sin(x_0-\alpha)=2\sin x_0=2\sin\frac{\alpha}{2}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.