2019 NSW ESA Mathematics Extension 2 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	A	C	D	B	B	D	C	D	A

Q1 $(3-2i)^2 = 5-12i$ **A**

Q2 $\int \frac{\sin x}{\cos^3 x}dx = \int -\frac{d(\cos x)}{\cos^3 x} = \frac{1}{2\cos^2 x} + c = \frac{1}{2}\sec^2 x + c$ **A**

Q3 $\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + c$ **C**

Q4 $1+(-3)+(-3) = -\frac{b}{2}$, $b = 10$ **D**

Q5 Sum of distances of z from foci $(-1,0)$ and $(2,0)$ equals 7. **B**

Q6 $2\sin|x| = y^2 \geq 0$, the graph of $y^2 = 2\sin|x|$ is undefined for $-2\pi < x < -\pi$ or $\pi < x < 2\pi$, and shows symmetry when reflected in both axes. **B**

Q7 For $0 \leq x \leq \frac{\pi}{4}$, $0 \leq \tan x \leq 1$

.: $\tan^2 x \leq \tan x$ and $1-\tan^2 x \geq 1-\tan x$

.: $\int_0^{\frac{\pi}{4}} \tan^2 x dx < \int_0^{\frac{\pi}{4}} \tan x dx$ and $\int_0^{\frac{\pi}{4}} 1-\tan^2 x dx > \int_0^{\frac{\pi}{4}} 1-\tan x dx$

Since $\int_0^{\frac{\pi}{4}} \tan x dx < \frac{1}{2}\left(\frac{\pi}{4}\right)$.: $\int_0^{\frac{\pi}{4}} 1-\tan x dx > \frac{1}{2}\left(\frac{\pi}{4}\right)$

.: $\int_0^{\frac{\pi}{4}} 1-\tan^2 x dx > \int_0^{\frac{\pi}{4}} 1-\tan x dx > \int_0^{\frac{\pi}{4}} \tan x dx > \int_0^{\frac{\pi}{4}} \tan^2 x dx$ **D**

Q8 $z^2 = -i\bar{z}$, let $z = r\text{cis}\theta$

$r^2\text{cis}2\theta = \text{cis}\left(-\frac{\pi}{2}\right) \times r\text{cis}(-\theta)$ and consider $r \neq 0$

$r^2\text{cis}2\theta - r\text{cis}\left(-\frac{\pi}{2}-\theta\right) = 0$, $r\left(r\text{cis}2\theta - \text{cis}\left(-\frac{\pi}{2}-\theta\right)\right) = 0$

.: $r\text{cis}2\theta = \text{cis}\left(-\frac{\pi}{2}-\theta\right)$, .: $r = 1$ and $2\theta = -\frac{\pi}{2}-\theta$, .: $\theta = -\frac{\pi}{6}$

.: $z = \text{cis}\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ **C**

Q9 $y = f(x)$ and $\frac{dy}{dx} = f'(x)$

For the inverse, $g'(x) = \frac{dx}{dy}$ (by interchanging x and y)

.: $g'(x) = \frac{1}{\frac{dy}{dx}} = \frac{1}{f'(x)}$ **D**

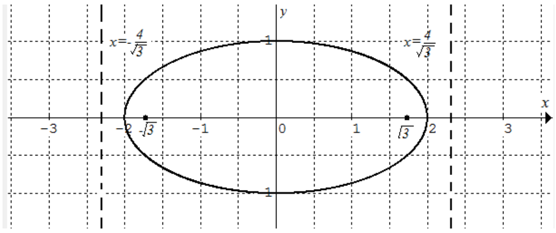
Q10 There are ${}^{10}C_2$ ways of selecting 2 digits from 10 digits, e.g. 3 and 7 are selected. For this selection there are 2^4 possible 4 digit codes with no restrictions.
Two of them have four of the same digit, 3333 and 7777.
.: Total number of codes using exactly 2 different digits is ${}^{10}C_2(2^4 - 2) = 630$ **A**

Section II

Q11ai $z + \bar{w} = (1+3i) + (2+i) = 3+4i$

Q11aii $\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{(1+3i)(2+i)}{(2-i)(2+i)} = \frac{-1+7i}{5} = -\frac{1}{5} + i\frac{7}{5}$

Q11b



Q11c $\frac{1}{2}\int \frac{2}{(x+5)^2+4}dx = \frac{1}{2}\tan^{-1}\left(\frac{x+5}{2}\right) + c$

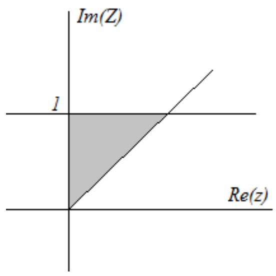
Q11d $\int \frac{6}{(x-3)(x+3)}dx$

$= \int \left(\frac{1}{x-3} - \frac{1}{x+3}\right)dx = \log_e|x-3| - \log_e|x+3| + c = \log_e\left|\frac{x-3}{x+3}\right| + c$

Q11ei $z = 2\text{cis}\frac{2\pi}{3}$

Q11eii $z^3 = 2^3\text{cis}\left(3 \times \frac{2\pi}{3}\right) = 8 + i0$

Q12a



Q12bi The cosine rule:

$x^2 + y^2 - 2xy\cos\frac{2\pi}{3} = 70^2$, $x^2 + xy + y^2 = 70^2$

Q12bii $2x\frac{dx}{dt} + y\frac{dx}{dt} + x\frac{dy}{dt} + 2y\frac{dy}{dt} = 0$

When $x = 30$ and $y = 50$ and $\frac{dy}{dt} = 4$

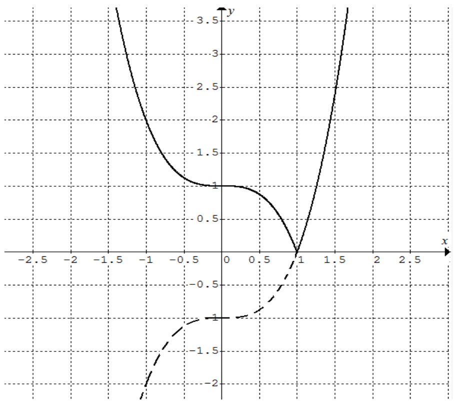
$110\frac{dx}{dt} = -130 \times 4$, $\frac{dx}{dt} = -\frac{52}{11}$ m s^{-1}

Q12c Area of curved cylindrical surface $= 2\pi x D(x) = 2\pi A x e^{-kx}$

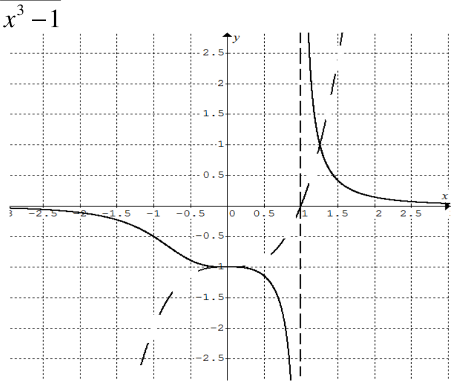
$\text{Vol} = 2\pi A\int_{10}^{40} xe^{-kx}dx = 2\pi A\left[\frac{xe^{-kx}}{-k} - \frac{e^{-kx}}{k^2}\right]_{10}^{40} = 2\pi A\left[\frac{e^{-kx}}{k^2}(-kx-1)\right]_{10}^{40}$

$= \frac{2\pi A}{k^2}\left(-e^{-40k}(40k+1) + e^{-10k}(10k+1)\right)$ km^3

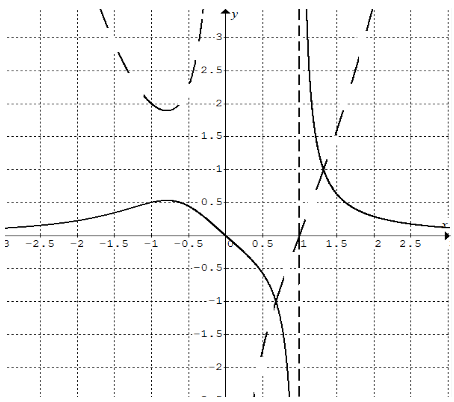
Q12di $y = |x^3 - 1|$



Q12dii $y = \dfrac{1}{x^3 - 1}$



Q12diii $y = \dfrac{x}{x^3 - 1} = \dfrac{1}{x^2 - \frac{1}{x}}$, sketch the reciprocal of $y = x^2 - \dfrac{1}{x}$



Q13ai $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $\dfrac{2x}{a^2} - \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = \dfrac{b^2 x}{a^2 y} = \dfrac{b\sec\theta}{a\tan\theta}$ at P

Tangent at P: $y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$, .: $bx\sec\theta - ay\tan\theta = ab$

Q13aii Similarly, tangent at Q: $bx\sec\phi - ay\tan\phi = ab$

.: at T, $bx_o\sec\theta - ay_o\tan\theta = ab$ and $bx_o\sec\phi - ay_o\tan\phi = ab$

Eliminating y_o: $bx_o(\sec\phi\tan\theta - \sec\theta\tan\phi) = ab(\tan\theta - \tan\phi)$

.: $x_o = \dfrac{a(\tan\theta - \tan\phi)}{\sec\phi\tan\theta - \sec\theta\tan\phi}$

Q13aiii $M\left(\dfrac{a(\sec\theta + \sec\phi)}{2}, \dfrac{b(\tan\theta + \tan\phi)}{2}\right)$,

gradient of $OM = \dfrac{b(\tan\theta + \tan\phi)}{a(\sec\theta + \sec\phi)}$

Given $y_o = \dfrac{b(\sec\theta - \sec\phi)}{\sec\phi\tan\theta - \sec\theta\tan\phi}$,

gradient of $OT = \dfrac{b(\sec\theta - \sec\phi)}{a(\tan\theta - \tan\phi)}$

Since $\tan^2\theta - \tan^2\phi = \sec^2\theta - \sec^2\phi$

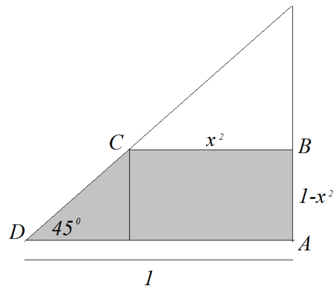
.: $(\tan\theta - \tan\phi)(\tan\theta + \tan\phi) = (\sec\theta - \sec\phi)(\sec\theta + \sec\phi)$

.: $\dfrac{(\tan\theta + \tan\phi)}{(\sec\theta + \sec\phi)} = \dfrac{(\sec\theta - \sec\phi)}{(\tan\theta - \tan\phi)}$

.: gradient of OM = gradient of OT

O is common for both line segments, .: O, T and M are collinear.

Q13bi Area of $ABCD = \dfrac{1}{2}(1 + x^2)(1 - x^2) = \dfrac{1 - x^4}{2}$



Q13bii $\text{Vol} = 2\times\int_0^1 \dfrac{1 - x^4}{2}dx = \dfrac{4}{5}$

Q13ci $y = -4.9t^2 + 10\sqrt{3}t = 0$, $t(-4.9t + 10\sqrt{3}) = 0$, $t = \dfrac{10\sqrt{3}}{4.9}$ s

and $x = 20\times\dfrac{10\sqrt{3}}{4.9}\cos\dfrac{\pi}{3} = \dfrac{100\sqrt{3}}{4.9}$ m

Q13cii Object 2:

Landing time $t = \dfrac{10\sqrt{3}}{4.9}$, $v\left(\dfrac{10\sqrt{3}}{4.9} - 2\right)\cos\phi = \dfrac{100\sqrt{3}}{4.9}$ and

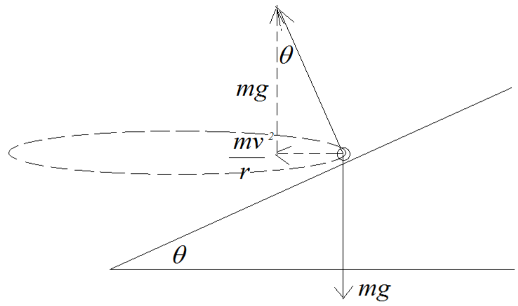
$-4.9\left(\dfrac{10\sqrt{3}}{4.9} - 2\right)^2 + v\left(\dfrac{10\sqrt{3}}{4.9} - 2\right)\sin\phi = 0$

.: $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{4.9\left(\frac{10\sqrt{3}}{4.9} - 2\right)^2}{\frac{100\sqrt{3}}{4.9}}$, .: $\phi \approx 0.3156$ or $18.0838°$

$\phi \approx 0.3$ or $18.1°$

$v\left(\dfrac{10\sqrt{3}}{4.9} - 2\right)\cos 0.3156 = \dfrac{100\sqrt{3}}{4.9}$, $v \approx 24.2$ m s^{-1}

Q14a $\tan\theta = \dfrac{\frac{mv^2}{r}}{mg}$, $v^2 = g\,r\tan\theta$



Q14bi $\ddot{x} = 0$ when terminal velocity is reached, $g - kw = 0$, $w = \dfrac{g}{k}$

Q14bii At $t = 0$, $v = 1.6w$ and $\dfrac{dv}{dt} = g - kv$,

$$\frac{dt}{dv} = \frac{1}{k\left(\frac{g}{k} - v\right)} = \frac{1}{k(w - v)} = \frac{-1}{k(v - w)} \text{ where } v - w > 0$$

$$kt = \int_{1.6w}^{1.1w} \frac{-1}{v - w}\,dv = \int_{1.1w}^{1.6w} \frac{1}{v - w}\,dv = \left[\log_e(v - w)\right]_{1.1w}^{1.6w} = \log_e 6$$

$\therefore\ t = \dfrac{1}{k}\log_e 6$ seconds

Q14biii $v\dfrac{dv}{dx} = g - kv$, $\dfrac{dv}{dx} = \dfrac{g - kv}{v} = \dfrac{-k(v - w)}{v}$, $\dfrac{dx}{dv} = \dfrac{-1}{k}\dfrac{v}{v - w}$

$$\frac{dx}{dv} = \frac{-1}{k}\left(1 + \frac{w}{v - w}\right),\ kD = \int_{1.6w}^{1.1w} -\left(1 + \frac{w}{v - w}\right)dv = \int_{1.1w}^{1.6w}\left(1 + \frac{w}{v - w}\right)dv$$

$$kD = \left[v + w\log_e(v - w)\right]_{1.1w}^{1.6w}$$

$$kD = 1.6w + w\log_e 0.6w - 1.1w - w\log_e 0.1w = w\left(\frac{1}{2} + \log_e 6\right)$$

$$\therefore\ D = \frac{g}{k^2}\left(\frac{1}{2} + \log_e 6\right)$$

Q14ci $\cot 2x + \operatorname{cosec} 2x = \dfrac{\cos 2x + 1}{\sin 2x} = \dfrac{2\cos^2 x}{2\sin x\cos x} = \cot x$

$\therefore\ \cot x - \cot 2x = \operatorname{cosec} 2x$

Q14cii $n = 1$, $\operatorname{cosec} 2x = \cot x - \cot 2x$ is true

Assume it is true for $n = k$, $\sum_{r=1}^{k} \operatorname{cosec} 2^r x = \cot x - \cot 2^k x$

For $n = k + 1$,

$$\sum_{r=1}^{k+1} \operatorname{cosec} 2^r x = \cot x - \cot 2^k x + \operatorname{cosec} 2^{k+1} x$$

$$= \cot x - \cot 2^k x + \operatorname{cosec} 2\left(2^k x\right)$$

$$= \cot x - \cot 2^k x + \cot 2^k x - \cot 2\left(2^k x\right)$$

$= \cot x - \cot 2^{k+1} x$ is also true

$\therefore\ \sum_{r=1}^{n} \operatorname{cosec} 2^r x = \cot x - \cot 2^n x$ is true for all $n \ge 1$

Q15ai Let $u = -x$, $\displaystyle\int_{-a}^{a} \frac{f(-x)}{f(x) + f(-x)}dx = \int_{a}^{-a} \frac{-f(u)}{f(-u) + f(u)}du$

$$= \int_{-a}^{a} \frac{f(u)}{f(-u) + f(u)}du = \int_{-a}^{a} \frac{f(x)}{f(-x) + f(x)}dx$$

Q15aii

$$\int_{-1}^{1} \frac{e^x}{e^x + e^{-x}}dx + \int_{-1}^{1} \frac{e^x}{e^x + e^{-x}}dx = \int_{-1}^{1} \frac{e^x}{e^x + e^{-x}}dx + \int_{-1}^{1} \frac{e^{-x}}{e^x + e^{-x}}dx$$

$$\therefore\ 2\int_{-1}^{1} \frac{e^x}{e^x + e^{-x}}dx = \int_{-1}^{1} dx = 2,\ \int_{-1}^{1} \frac{e^x}{e^x + e^{-x}}dx = 1$$

Q15bi When $w = y$,

$$\Pr(A \text{ winning}) = \frac{1}{2} + \frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3} > \frac{1}{2}$$

Q15bii $\Pr(A \text{ winning}) < \dfrac{1}{2}$

$$\frac{w}{w + y} + \left(\frac{y}{w + y}.\frac{w}{w + y}\right)\frac{w}{w + y} + \left(\frac{y}{w + y}.\frac{w}{w + y}\right)\left(\frac{y}{w + y}.\frac{w}{w + y}\right)\frac{w}{w + y} + \ldots < \frac{1}{2}$$

$$\frac{\frac{w}{w + y}}{1 - \frac{wy}{(w + y)^2}} < \frac{1}{2},\ \frac{w(w + y)}{(w + y)^2 - wy} < \frac{1}{2},\ \left(\frac{y}{w}\right)^2 - \frac{y}{w} - 1 > 0$$

$$\therefore\ \frac{y}{w} > \frac{1 + \sqrt{5}}{2}$$

Q15ci Let $u = x + 1$,

$$\int_0^1 \frac{x}{(x + 1)^2}dx = \int_1^2 \frac{u - 1}{u^2}du = \int_1^2 \frac{1}{u} - \frac{1}{u^2} = \left[\ln u + \frac{1}{u}\right]_1^2 = \ln 2 - \frac{1}{2}$$

Q15cii Let $u = x^n$ and $\dfrac{dv}{dx} = \dfrac{1}{(x + 1)^2}$, $\therefore\ \dfrac{du}{dx} = nx^{n-1}$ and $v = \dfrac{-1}{x + 1}$.

$$I_n = \int_0^1 \frac{x^n}{(x + 1)^2}dx = \int_0^1 u\frac{dv}{dx}dx = [uv]_0^1 - \int_0^1 v\frac{du}{dx}dx$$

$$= \left[\frac{-x^n}{x + 1}\right]_0^1 - \int_0^1 \frac{-nx^{n-1}}{x + 1}dx = \left[\frac{-x^n}{x + 1}\right]_0^1 + \int_0^1 \frac{nx^{n-1}}{x + 1}dx$$

$$= \left[\frac{-x^n}{x + 1}\right]_0^1 + n\int_0^1 \frac{x^{n-1}(x + 1)}{(x + 1)^2}dx = \left[\frac{-x^n}{x + 1}\right]_0^1 + n\int_0^1 \frac{x^n}{(x + 1)^2} + \frac{x^{n-1}}{(x + 1)^2}dx$$

$$\therefore\ I_n = -\frac{1}{2} + nI_n + nI_{n-1},\ I_n = \frac{1}{2(n - 1)} - \frac{n}{n - 1}I_{n-1},\ n \ge 2$$

Q15ciii $I_1 = \ln 2 - \dfrac{1}{2}$, $I_2 = \dfrac{1}{2} - 2I_1 = \dfrac{3}{2} - 2\ln 2$

$$I_3 = \frac{1}{4} - \frac{3}{2}I_2 = 3\ln 2 - 2$$

Q16ai $x^3 - px + q = (r\cos\theta)^3 - p(r\cos\theta) + q$

$= r^3\cos^3\theta - \frac{3r^2}{4}(r\cos\theta) + q = r^3\cos^3\theta - \frac{3r^3\cos\theta}{4} + q$

$= \frac{r^3}{4}\left(4\cos^3 - 3\cos\theta + \frac{4q}{r^3}\right) = \frac{r^3}{4}\left(\cos 3\theta + \frac{4q}{r^3}\right) = 0$

$\therefore$ $r\cos\theta$ is a root of $x^3 - px + q = 0$

Q16aii $x^3 + 9x^2 + 15x - 17 = (x-\alpha)(x-\beta)(x-\gamma)$

$(x-3)^3 + 9(x-3)^2 + 15(x-3) - 17 = (x-3-\alpha)(x-3-\beta)(x-3-\gamma)$

$x^3 - 12x - 8 = (x-(\alpha+3))(x-(\beta+3))(x-(\gamma+3))$

$\therefore$ $\alpha+3$, $\beta+3$, $\gamma+3$ are the roots of $x^3 - 12x - 8 = 0$

Q16aiii $p = 12$, $q = -8$ and $r = 4$,

$\therefore$ $\cos 3\theta = \frac{1}{2}$, $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

The roots of $x^3 - 12x - 8 = 0$ are $4\cos\frac{\pi}{9}$, $4\cos\frac{5\pi}{9}$, $4\cos\frac{7\pi}{9}$

$\therefore$ the roots of $x^3 + 9x^2 + 15x - 17 = 0$ are

$4\cos\frac{\pi}{9} - 3$, $4\cos\frac{5\pi}{9} - 3$, $4\cos\frac{7\pi}{9} - 3$

Q16bi $\alpha = \text{cis}\theta$, $\beta = \text{cis}\phi$, $\overline{\alpha} = \text{cis}(-\theta)$, $\overline{\beta} = \text{cis}(-\phi)$

$\alpha + \overline{\alpha} + \beta + \overline{\beta} = 2k$, $(\alpha + \overline{\alpha} + \beta + \overline{\beta})^2 = (2k)^2$,

$\alpha^2 + \overline{\alpha}^2 + \beta^2 + \overline{\beta}^2 + 2(\alpha\overline{\alpha} + \beta\overline{\beta} + \alpha\beta + \alpha\overline{\beta} + \overline{\alpha}\beta + \overline{\alpha}\overline{\beta}) = 4k^2$

Since $\alpha\overline{\alpha} + \beta\overline{\beta} + \alpha\beta + \alpha\overline{\beta} + \overline{\alpha}\beta + \overline{\alpha}\overline{\beta} = 2k^2$

$\therefore$ $\alpha^2 + \overline{\alpha}^2 + \beta^2 + \overline{\beta}^2 = 0$

$\text{cis}2\theta + \text{cis}(-2\theta) + \text{cis}2\phi + \text{cis}(-2\phi) = 0$

$\therefore$ $2\cos 2\theta + 2\cos 2\phi = 0$, $2(\cos 2\theta + \cos 2\phi) = 0$

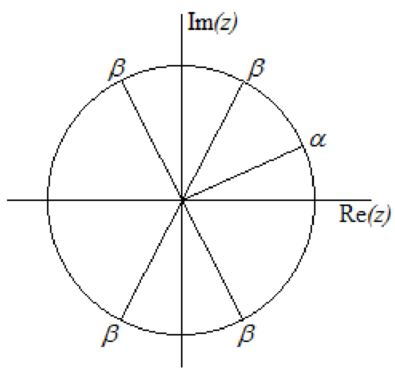
$2(2\cos^2\theta - 1 + 2\cos^2\phi - 1) = 0$,

$2(\text{Re}(\alpha))^2 + 2(\text{Re}(\beta))^2 - 2 = 0$, $\therefore$ $(\text{Re}(\alpha))^2 + (\text{Re}(\beta))^2 = 1$

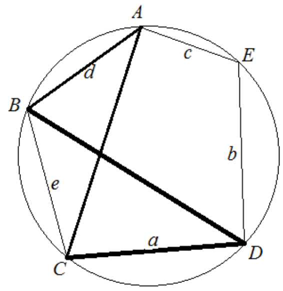
Q16bii $(\text{Re}(\alpha))^2 + (\text{Im}(\alpha))^2 = 1$ and $(\text{Re}(\beta))^2 + (\text{Im}(\beta))^2 = 1$

$\therefore$ $(\text{Re}(\beta))^2 = (\text{Im}(\alpha))^2$ and $(\text{Im}(\beta))^2 = (\text{Re}(\alpha))^2$

$\therefore$ $\text{Re}(\beta) = \pm\text{Im}(\alpha)$ and $\text{Im}(\beta) = \pm\text{Re}(\alpha)$



Q16c



$\frac{e}{\sin\angle BDC} = \frac{a}{\sin(\angle ABC - \angle ABD)}$

$= \frac{a}{\sin(\angle ABC - (180° - \angle AED))}$ ($ABDE$ is a cyclic quadrilateral)

$= \frac{a}{\sin(-(180° - (\angle ABC + \angle AED)))}$

$= \frac{a}{-\sin(180° - (\angle ABC + \angle AED))}$

$= \frac{a}{-\sin(\angle ABC + \angle AED)}$

$= \frac{a}{-\sin(B+E)}$

$\frac{e}{\sin\angle BAC} = \frac{d}{\sin(\angle BCD - \angle ACD)}$

$= \frac{d}{\sin(\angle BCD - (180° - \angle AED))}$ ($ACDE$ is a cyclic quadrilateral)

$= \frac{d}{\sin(-(180° - (\angle BCD + \angle AED)))}$

$= \frac{d}{-\sin(180° - (\angle BCD + \angle AED))}$

$= \frac{d}{-\sin(\angle BCD + \angle AED)}$

$= \frac{d}{-\sin(C+E)}$

Since $\angle BDC = \angle BAC$ (subtended by the same chord)

$\therefore$ $\frac{a}{\sin(B+E)} = \frac{d}{\sin(C+E)}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.