2019 NSW ESA Mathematics Exam Solutions

Section I

1	2	3	4	5	6	7	8	9	10
B	D	B	C	A	C	D	C	A	C

Q1 **B**

Q2 **D**

Q3 **B**

Q4 **C**

Q5 $\frac{\log_2 9}{\log_2 3} = \frac{2\log_2 3}{\log_2 3} = 2$ **A**

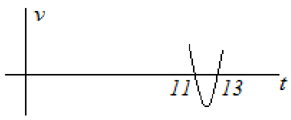
Q6 Pr $\Pr(winning) = \Pr(ET) + \Pr(EH) + \Pr(OT) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$ **C**

Q7 $\frac{2\pi}{b} = \pi$ **D**

Q8 Velocity increases from $t = 0$ to $t = 3$ and decreases from $t = 3$ to $t = 5$. **C**

Q9 $\int \sec^2 x - 1\, dx = \tan x - x + C$ **A**

Q10 A possible velocity-time graph is shown below. **C**



Section II

Q11a $\frac{x}{\sin 40°} = \frac{8}{\sin 110°}$, $x = 5.5$

Q11b $\frac{d}{dx} x^2 \sin x = 2x\sin x + x^2 \cos x$

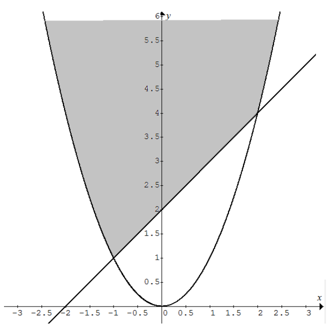
Q11c $\frac{d}{dx}\left(\frac{2x+1}{x+5}\right) = \frac{d}{dx}\left(2 - \frac{9}{x+5}\right) = \frac{9}{(x+5)^2}$

Q11d $a = 2000$, $r = -0.6$, $S_\infty = \frac{2000}{1-(-0.6)} = 1250$

Q11e $\int_0^1 \frac{1}{(3x+2)^2} dx = \left[\frac{-1}{3(3x+2)}\right]_0^1 = -\frac{1}{15} + \frac{1}{6} = \frac{1}{10}$

Q11f $\Pr(\text{2 the same colour}) = \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{6}{11} = \frac{31}{66}$

Q11g



Q12ai Gradient of $\ell = \frac{1}{2}$, line AC $\frac{y-6}{x-8} = -2$, $y = -2x + 22$

Q12aii $B(-4,0)$, $C = (11,0)$, area $\Delta ABC = \frac{1}{2}(11-(-4)) \times 6 = 45$

Q12b Solve the simultaneous equations $t_4 = a + 3d = 6$ and $S_{16} = 8(2a + 15d) = 120$, $\therefore$ $d = \frac{1}{3}$

Q12ci $L(31) = 200\,000 e^{-0.14 \times 31} \approx 2607$

Q12cii $L'(31) = -0.14 \times 200000 e^{-0.14 \times 31} \approx -365$ i.e. dropping 365 leaves per day when $t = 31$

Q12ciii $200000 e^{-0.14t} = 100$, $-0.14t = \log_e\left(\frac{100}{200000}\right)$, $t \approx 54.3$

Q12d Area $= \int_0^3 \frac{\frac{3}{2} \times 2x}{x^2+1} dx = \frac{3}{2}[\log_e u]_1^{10} = \frac{3}{2}\log_e 10$ where $u = x^2 + 1$.

Q13a $2\sin x \cos x - \sin x = 0$, $\sin x(2\cos x - 1) = 0$ and $0 \le x \le 2\pi$

$\therefore$ $\sin x = 0$ or $\cos x = \frac{1}{2}$ $\therefore$ $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

Q13b Perimeter $=$ arc AB $+$ chord AB

$= \frac{70}{360} \times 2\pi \times 20 + 2 \times 20 \sin 35° \approx 47.4\,\text{cm}$

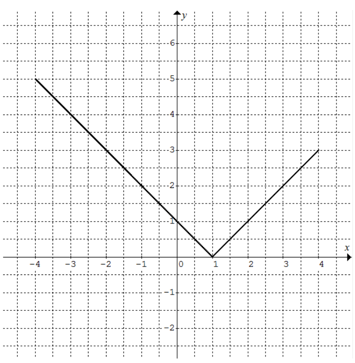
Q13ci $\frac{d}{dx}(\ln x)^2 = 2(\ln x)\frac{1}{x} = \frac{2\ln x}{x}$

Q13cii $\int \frac{2\ln x}{x} dx = (\ln x)^2 + c$, $\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 + C$

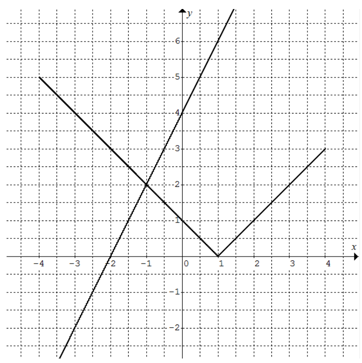
Q13d Volume $= \int_0^1 \pi(x - x^3)^2 dx = \pi\int_0^1 x^2 - 2x^4 + x^6\, dx$

$= \pi\left[\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7}\right]_0^1 = \frac{8\pi}{105}$

Q13ei



Q13eii $x = -1$



Q14a $t=0$, $v=0$, $\frac{dv}{dt}=e^{2t}-4$.: $v=\int e^{2t}-4\,dt=\frac{1}{2}e^{2t}-4t-\frac{1}{2}$

Q14bi Let $f'(x)=3x^2+2x-1=(3x-1)(x+1)=0$

.: stationary points are at $x=-1$, $x=\frac{1}{3}$

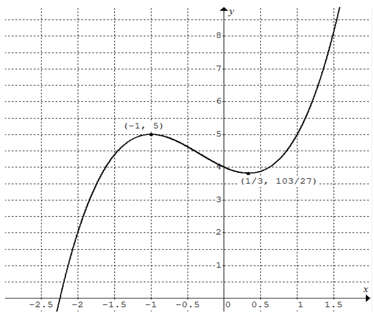
Since $f'(x)>0$ for $x>\frac{1}{3}$, $f'(x)<0$ for $-1<x<\frac{1}{3}$ and $f'(x)>0$ for $x<-1$, .: the stationary point at $x=-1$ is a local maximum and the stationary point at $x=\frac{1}{3}$ is a local minimum.

Q14bii $(0,4)$ is a point of $f(x)$,

.: $f(x)=\int 3x^2+2x-1\,dx=x^3+x^2-x+4$

Q14biii

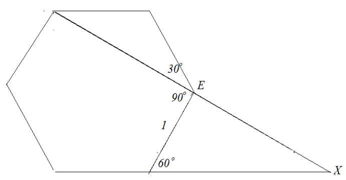
$x=-1$, $y=f(-1)=5$, $x=\frac{1}{3}$, $y=\frac{103}{27}$



Q14biv The point of inflection is at $x=\frac{-1+\frac{1}{3}}{2}=-\frac{1}{3}$

The curve is concave down for $x<-\frac{1}{3}$.

Q14c $EX=\tan 60°=\sqrt{3}$



Q14d $y=x^3+ax^2+bx+4=x-4$ at $x=2$, .: $2a+b=-7$

$\frac{dy}{dx}=3x^2+2ax+b=1$ at $x=2$, .: $4a+b=-11$

Solve simultaneous equations, $a=-2$, $b=-3$

Q15a $e^{2\ln x}=x+6$ and $x>0$, $e^{\ln(x^2)}=x+6$, $x^2=x+6$

$x^2-x-6=0$, .: $x=3$

Q15b $\overline{AC}^2+\overline{BC}^2=\overline{AB}^2$, .: $p^2+h^2+h^2+q^2=(p+q)^2$

.: $h^2=pq$, $h=\sqrt{pq}$

Q15ci Let $\overline{BR}=a$, $\frac{a}{8}=\frac{1}{x}$ (similar triangles), $a=\frac{8}{x}$

$D^2=(x+8)^2+(a+1)^2$, .: $D^2=(x+8)^2+\left(\frac{8}{x}+1\right)^2$

Q15cii Let $\frac{d}{dx}D^2=2(x+8)+2\left(\frac{8}{x}+1\right)\left(-\frac{8}{x^2}\right)=0$

.: $(x+8)\left(1-\frac{8}{x^3}\right)=0$, .: $x=2$

For $0<x<2$, $\frac{d}{dx}D^2<0$, for $x>2$, $\frac{d}{dx}D^2>0$

.: $x=2$ gives the minimum value of D^2.

Q15di $\Pr(\text{at least one})=1-\Pr(\text{none})=1-0.98^2=0.0396$

Q15dii $\Pr(\text{at least one})=1-\Pr(\text{none})=1-0.98^n>0.4$

.: $0.98^n<0.6$, $n>25.285$ (3 dec. places), .: the smallest n is 26.

Q16ai $A_1=1000000\left(1+\frac{6}{4\times 100}\right)^4-80000=1000000\times 1.015^4-80000$

$A_2=(1000000\times 1.015^4-80000)\times 1.015^4-80000$

$=1000000\times 1.015^8-80000\times 1.015^4-80000$

$=1000000\times 1.015^8-80000(1+1.015^4)$

Q16aii $A_3=1000000\times 1.015^{4\times 3}-80000(1+1.015^{4\times 1}+1.015^{4\times 2})$

$A_n=1000000\times 1.015^{4n}-80000(1+1.015^{4\times 1}+1.015^{4\times 2}+...+1.015^{4(n-1)})>0$

$1000000\times 1.015^{4n}-80000\left(\frac{(1.015^4)^n-1}{1.014^4-1}\right)>0$, $n\le 24$

The full amount of $\$80000$ can be withdrawn for 24 years.

Q16b $t=0$, $x=0$, $v=e^{\cos t}-1$

When $v=0$, $e^{\cos t}-1=0$, $\cos t=0$, $t=\frac{\pi}{2}$

Estimation $=\frac{\frac{\pi}{2}}{6}\left((e-1)+4\left(e^{\cos\frac{\pi}{4}}-1\right)+0\right)\approx 1.53$

Particle's position is 1.53 m from the origin.

Q16ci $\frac{dy}{dx}=rx^{r-1}=r$ at $x=1$, equation of tangent is $\frac{y-1}{x-1}=r$.

x-intercept: $\frac{-1}{x-1}=r$, $x=1-\frac{1}{r}=\frac{r-1}{r}$ $\left(\frac{r-1}{r},0\right)$

Q16cii Area of region $R=A=\int_0^1 x^r\,dx-\frac{1}{2}\left(1-\frac{r-1}{r}\right)$

$=\left[\frac{x^{r+1}}{r+1}\right]_0^1-\frac{1}{2r}=\frac{1}{r+1}-\frac{1}{2r}=\frac{r-1}{2r(r+1)}$

Q16ciii Let $\frac{dA}{dr}=\frac{2r(r+1)-(r-1)(4r+2)}{4r^2(r+1)^2}=\frac{-2(r^2-2r-1)}{4r^2(r+1)^2}=0$

.: $r=1+\sqrt{2}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.