2019 Specialist Mathematics Trial Exam 1 Solutions

Q1a As $x \to \pm\infty$, $f(x) \to \frac{x^2}{4}$; as $x \to 0^{\pm}$, $f(x) \to -\infty$

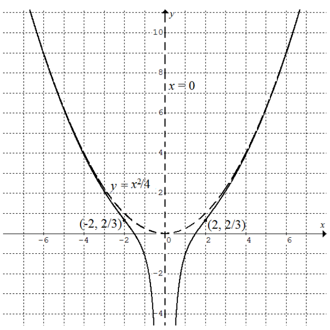
Let $y = f(x) = 0$, $\frac{x^2}{4} - \frac{4}{3x^2} = 0$, $x^4 = \frac{16}{3}$, $x = \pm\frac{2}{\sqrt[4]{3}}$.: $\left(\pm\frac{2}{\sqrt[4]{3}}, 0\right)$

Let $f'(x) = \frac{x}{2} + \frac{8}{3x^3} = \frac{3x^4 + 16}{6x^3}$, $x \neq 0$ and $3x^4 + 16 > 0$

.: $f'(x) \neq 0$, no stationary points. No y-intercept.

Let $f''(x) = \frac{1}{2} - \frac{8}{x^4} = 0$.: $x = \pm 2$, $y = \frac{2}{3}$.: $\left(\pm 2, \frac{2}{3}\right)$ inflection pts

Q1b



Q2 $u = 1 + e^{x^2}$, $\frac{du}{dx} = 2xe^{x^2}$,

$\int_0^1 \left(\frac{\frac{1}{2}\frac{du}{dx}}{u}\right) dx = \frac{1}{2}\int_2^{1+e} \frac{1}{u} du = \frac{1}{2}[\log_e u]_2^{1+e} = \frac{1}{2}\log_e\left(\frac{1+e}{2}\right)$

Q3a $-\frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2} = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{2}\,\text{cis}\left(\frac{5\pi}{6}\right)$

$\left(\sqrt{2}\,\text{cis}\left(\frac{5\pi}{6}\right)\right)^6 + n = 0$, $8\,\text{cis}(5\pi) + n = 0$.: $n = 8$

Q3b The conjugate root is $-\frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}$, the other 4 roots equally space out on a circle of radius $\sqrt{2}$, they are $\pm\sqrt{2}\,i$, $\frac{\sqrt{6}}{2} \pm i\frac{\sqrt{2}}{2}$.

Q4a $\tilde{p}.\tilde{q} = 0$, $-2\alpha - \sqrt{3}\beta = 0$, $\frac{\alpha}{\beta} = -\frac{\sqrt{3}}{2}$

Q4b $\tilde{p}.\tilde{j} = |\tilde{p}||\tilde{j}|\cos\theta$, $-2 = 3 \times 1\cos\theta$, $\cos\theta = -\frac{2}{3}$

Q4c Let $\tilde{r} = \tilde{p} + \tilde{q} = \sqrt{2}\,\tilde{i} - 2\,\tilde{j} + \sqrt{3}\,\tilde{k} + \left(-\frac{\sqrt{3}\,\beta}{2}\right)\tilde{j} - \beta\tilde{k}$

$= \sqrt{2}\,\tilde{i} - \left(2 + \frac{\sqrt{3}\,\beta}{2}\right)\tilde{j} + \left(\sqrt{3} - \beta\right)\tilde{k}$

Q5 $\int \frac{y}{y^2 - 1} dy = -\int \frac{x}{x^2 - 1} dx$, let $u = x^2 - 1$ and $v = y^2 - 1$

.: $\int \frac{1}{v} dv = -\int \frac{1}{u} du$, $\log_e|v| = -\log_e|u| + c$,

$\log_e|y^2 - 1| = -\log_e|x^2 - 1| + c$, $\log_e\left|(y^2 - 1)(x^2 - 1)\right| = c$.

Given that $(-2, 2)$ satisfies the relation .: $c = \log_e 9$.

.: $y = \sqrt{\frac{9}{x^2 - 1} + 1}$

Q6 $\overrightarrow{OM} = \frac{1}{2}\left(\tilde{a} + \tilde{b}\right)$, $\overrightarrow{ON} = \frac{4}{5}\overrightarrow{OM} = \frac{2}{5}\left(\tilde{a} + \tilde{b}\right)$,

$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \frac{2}{5}\left(\tilde{a} + \tilde{b}\right) - \tilde{a} = \frac{1}{5}\left(2\tilde{b} - 3\tilde{a}\right)$

Q7a $\sin^2\left(\frac{\pi}{x}\right) - \cos^2\left(\frac{\pi}{x}\right) = \frac{1}{2}$, $-\sin^2\left(\frac{\pi}{x}\right) + \cos^2\left(\frac{\pi}{x}\right) = -\frac{1}{2}$,

$\cos 2\left(\frac{\pi}{x}\right) = -\frac{1}{2}$, $\frac{2\pi}{x} = \pm\frac{8\pi}{3}$, $x = \pm\frac{3}{4}$

Q7b The domain for $\cos^{-1}\left(-\frac{x}{\pi}\right)$ or $\sin^{-1}\left(\frac{x}{\pi}\right)$ is $x \in [-\pi, \pi]$.

$\cos^{-1}\left(-\frac{x}{\pi}\right)$ is the reflection of $\cos^{-1}\left(\frac{x}{\pi}\right)$ in the y-axis. Translating $\cos^{-1}\left(-\frac{x}{\pi}\right)$ in the positive y-direction by $\frac{\pi}{2}$ will make it coincide with $\sin^{-1}\left(\frac{x}{\pi}\right)$. $\sin^{-1}\left(\frac{x}{\pi}\right) = \cos^{-1}\left(-\frac{x}{\pi}\right) + \frac{\pi}{2}$ is an identity, i.e. it is true for $x \in [-\pi, \pi]$

Q8

$(x^2 - 1)\frac{dy}{dx} + 2xy = 2(y^2 - 1) + 4xy\frac{dy}{dx}$, $(x^2 - 1 - 4xy)\frac{dy}{dx} = 2(y^2 - 1 - xy)$,

$\frac{dy}{dx} = \frac{2(y^2 - 1 - xy)}{x^2 - 1 - 4xy}$, $\frac{dy}{dx} = \frac{1}{2}$ at all four points.

Q9 Horizontal component of reaction force $= 9.8\cos 60° = 4.9$

Vert. component of reaction force $= 1 \times 9.8 - 9.8\sin 60° = 4.9\left(2 - \sqrt{3}\right)$

.: $\tan\theta° = 2 - \sqrt{3}$

Q10a $\text{E}(Y) = 3\text{E}(X) + 30 = 195$, $\text{Var}(Y) = 3^2\,\text{Var}(X)$, $\sigma_Y = 3\sigma_X = 6$

Q10b The weights of individual nuts (or bolts) are independent random variables.

$\mu_W = \overline{W} = \text{E}(W) = \text{E}(X_1 + X_2 + Y_1 + Y_2) = \text{E}(X_1) + \text{E}(X_2) + \text{E}(Y_1) + \text{E}(Y_2) = 500$

$\text{Var}(W) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(Y_1) + \text{Var}(Y_2) = 4 + 4 + 36 + 36 = 80$

.: $\sigma_W = \sqrt{80}$

Q10c $\sigma_{\overline{W}} = \frac{\sigma_W}{\sqrt{n}} = \frac{\sqrt{80}}{\sqrt{100}} = \sqrt{0.8} \approx 0.9$

Q10d $(491, 509) \approx \left(\mu_{\overline{W}} - 10\sigma_{\overline{W}},\ \mu_{\overline{W}} + 10\sigma_{\overline{W}}\right)$,

.: $\Pr\left(491 < \overline{W} < 509\right) \approx 1$

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