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Specialist Mathematics

2019

Trial Examination 2
(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 The graph of $y = f(x)$ shows symmetry under reflection in the y-axis.
$f(x)$ could be

A. $1-\dfrac{1}{|1-x|}$

B. $\left|x-\dfrac{1}{|x|}\right|$

C. $\left|1-\dfrac{1}{x}\right|$

D. $1-\dfrac{1}{x}$

E. $\dfrac{1+|x|}{1-|x|}$

Question 2 Given $\sec(2\pi(x-a)) - \operatorname{cosec}(2\pi(x+a)) = 0$, the value of a could be

A. $\dfrac{1}{2}$

B. $-\dfrac{1}{3}$

C. $\dfrac{1}{4}$

D. $-\dfrac{1}{6}$

E. $\dfrac{1}{8}$

Question 3 $a\sin^{-1}(nx)+b\cos^{-1}(nx)=\frac{(a+b)\pi}{4}$ has infinitely many solutions for x when

A. $a+b=0$ and $-\frac{\pi}{n}\le x\le\frac{1}{n}$

B. $a-b=0$ and $-\frac{1}{n}\le x\le\frac{\pi}{n}$

C. $a+b=0$ and $-\frac{1}{n}\le x\le\frac{1}{n}$

D. $a-b=0$ and $-\frac{1}{n}\le x\le\frac{1}{n}$

E. $a+b=0$ and $-\frac{1}{\pi}\le nx\le\frac{1}{\pi}$

Question 4 The graph of $y=2\left(\sin^{-1}\left(\frac{x}{a}-1\right)-\frac{b}{2}\right)$, where $a>0$, is translated to the left by a units and then reflected in the x-axis. The point of inflection of the graph after the two transformations is

A. $(0, b)$

B. $(-a, b)$

C. $(1, -b)$

D. $(-1, b)$

E. $(b, -a)$

Question 5 The product of two of the four linear factors of $f(z)=z^4+3z^2+4$ could be

A. z^2+z-2

B. z^2-z+2

C. z^2+iz+2

D. z^2-iz-2

E. z^2-z-2i

Question 6 If $z_1 = \alpha \operatorname{cis}\left(\theta + \frac{2\pi}{3}\right)$, $z_2 = \beta \operatorname{cis}\left(\theta + \frac{\pi}{6}\right)$, where $\alpha, \beta \in R$ and $0 < \beta < \alpha$, then

A. $|z_1 - z_2| = \alpha + \beta$

B. $|z_2 - z_1| = \alpha - \beta$

C. $|z_1 - z_2| = \sqrt{\alpha^2 + \beta^2}$

D. $|z_2 - z_1| = \sqrt{\alpha + \beta}$

E. $|z_1 - z_2| = \sqrt{\alpha^2 - \beta^2}$

Question 7 $\{z : |z| - |z - 1| = a,\ a \in R\}$ is a hyperbola on an Argand plane.
Which one of the following statements is true?

A. $a = 1$

B. $a = -1$

C. $-1 \le a < 0$

D. $0 < a < 1$

E. $a > 1$

Question 8 Given $\arg\left(\frac{z-1}{z-i}\right) = \pi$ and $a \le |z| < 1$, the value of a is

A. $\frac{1}{\sqrt{2}}$

B. $\frac{1}{2}$

C. $\frac{1}{4}$

D. $\frac{\sqrt{3}}{2}$

E. $\frac{1}{3}$

Question 9 The length of the minor arc of a unit circle centred at $(0,0)$ for $-0.5 \le x \le -0.1$ is given by

A. $\displaystyle\int_{-0.5}^{-0.1} \sqrt{1-\frac{x}{\sqrt{1-x^2}}}\,dx$

B. $\displaystyle\int_{-0.1}^{-0.5} \sqrt{1-\frac{x^2}{1-x^2}}\,dx$

C. $\displaystyle\int_{-0.5}^{-0.1} \sqrt{1+\frac{4x^2}{1-x^2}}\,dx$

D. $\displaystyle\int_{-0.1}^{-0.5} \sqrt{1+\frac{2x^2}{1-x^2}}\,dx$

E. $\displaystyle\int_{-0.5}^{-0.1} \frac{1}{\sqrt{1-x^2}}\,dx$

Question 10 The area of the region bounded by the *x*-axis, the *y*-axis, the line $y=1$ and the curve $y=\frac{1}{\pi}\cos^{-1}(x-a)$ where $a>1$ is

A. π

B. a

C. $a\pi$

D. 2

E. $2a$

Question 11 The value of $\displaystyle\int_{-\sqrt{a}}^{\sqrt{a}} \frac{1}{\sqrt{a-x^2}}\,dx$ is

A. π

B. $2\sqrt{a}$

C. $a\pi$

D. 2

E. undefined

Question 12 The graph of $y = \dfrac{1}{\frac{1}{x^2} + \frac{1}{10x} + 1}$ has n points of inflection. The value of n is

A. 0

B. 1

C. 2

D. 3

E. 4

Question 13 $2\underset{\sim}{\mathrm{i}} - \underset{\sim}{\mathrm{j}}$, $\underset{\sim}{\mathrm{i}} - 2\underset{\sim}{\mathrm{j}} + \underset{\sim}{\mathrm{k}}$ and $a\underset{\sim}{\mathrm{i}} + b\underset{\sim}{\mathrm{j}} + 2\underset{\sim}{\mathrm{k}}$ are linear *independent* vectors.
Which one of the following statements **cannot** be true?

A. $a = 5$

B. $b = -2$

C. $a + 2b + 6 = 0$

D. $2a + b = 4$

E. $a + b + 3 = 0$

Question 14 The angle between vectors $a\underset{\sim}{\mathrm{i}} + b\underset{\sim}{\mathrm{j}} + \underset{\sim}{\mathrm{k}}$ and $\underset{\sim}{\mathrm{j}}$ is $120°$.
Which one of the following statements is true?

A. $b = \sqrt{\dfrac{1 + a^2}{3}}$

B. $b = -\sqrt{\dfrac{1 + a^2}{3}}$

C. $b = \sqrt{\dfrac{1 - a^2}{3}}$

D. $b = -\sqrt{\dfrac{1 - a^2}{3}}$

E. $b = \sqrt{\dfrac{a^2 - 1}{3}}$

Question 15 A particle has position vector $\underset{\sim}{r}(t)=\frac{2t}{t^2+1}\underset{\sim}{i}+\frac{t^2-1}{t^2+1}\underset{\sim}{j}$, $t \geq 0$.
Which one of the following statements gives the best description of the motion of the particle?

A. The particle moves in the anticlockwise direction along a circular path of radius 1 unit.

B. The particle moves in the clockwise direction along a circular path of radius 1 unit.

C. The particle moves in the anticlockwise direction along a semi-circular path.

D. The particle moves in the clockwise direction along an elliptical path.

E. The particle moves in the anticlockwise direction along an elliptical path.

Question 16 A spring balance is fastened to the ceiling of a lift with a string. An object is hooked to the spring balance. The lift slows down at 1.0 m s^{-2} while it moves downwards, and the spring balance reading is 5.4 N. The inertial mass (in kg) of the object is closest to

A. 0.50

B. 0.61

C. 1.00

D. 4.40

E. 5.40

Question 17 A crate is pulled by a cord of tension 100 N inclined at $60°$ to a horizontal plane. The crate moves on the horizontal plane against 30 N force of friction for 5 seconds.
The change of momentum (in kg m s^{-1}) of the crate is

A. 5

B. 10

C. 100

D. 150

E. 196

Question 18 Random variables X and Y are the weights (kg) of a 1-kg bag of apples and a 2-kg bag of oranges respectively in a supermarket chain.
$\text{E}(X)$ and $\text{Var}(X)$ are 1.10 and 0.0016 respectively; $\text{E}(Y)$ and $\text{Var}(Y)$ are 2.10 and 0.0025 respectively. Let random variable W be the total weight (kg) of 2 bags of apples and a bag of oranges from the supermarket chain.

$\Pr(W < 4.00)$ is closest to

A. 0.0000

B. 0.0002

C. 0.001

D. 0.002

E. 0.01

Question 19 Random variable X is the weight (kg) of 1-kg bag of apples in a supermarket chain.
$\text{E}(X)$ and $\text{Var}(X)$ are 1.10 and 0.0016 respectively.
A random sample of n 1-kg bags of apples from the supermarket chain is taken.

If $\Pr(\overline{X} < 1.092) \approx 0.025$, the value of n is closest to

A. 50

B. 100

C. 200

D. 400

E. 625

Question 20 Two years ago the mean height and the mean weight of an 18-year old population were 175 cm and 70 kg. Recently a random sample of 1200 people was taken from the same population (now 20-year old). The sample mean height and standard deviation were 176 cm and 22 cm respectively, and the sample mean weight and standard deviation were 71 kg and 22 kg respectively.

For the 20-year old population, there is **no** significant evidence to suggest that in the past two years

A. the mean height has **not** changed

B. the mean weight has **not** changed

C. both the mean height and the mean weight have **not** changed

D. both the mean height and the mean weight have changed

E. the mean height has changed and the mean weight has **not** changed

SECTION B Extended-answer questions

Instructions for Section B

Answer **all** questions.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 **Unit** vectors $\tilde{u}$ and $\tilde{v}$ make a $60°$ angle.
Vector $\tilde{u}$ makes $45°$, $60°$ and $120°$ with the orthogonal unit vectors $\tilde{i}$, $\tilde{j}$ and $\tilde{k}$ respectively.
$\tilde{v}$ has the same scalar resolutes in the $\tilde{j}$ and $\tilde{k}$ directions.

a. Show that $\tilde{u} = \frac{1}{2}\left(\sqrt{2}\,\tilde{i} + \tilde{j} - \tilde{k}\right)$. 1 mark

b. Hence show that $\tilde{v} = \frac{1}{2}\left(\sqrt{2}\,\tilde{i} + \tilde{j} + \tilde{k}\right)$. 2 marks

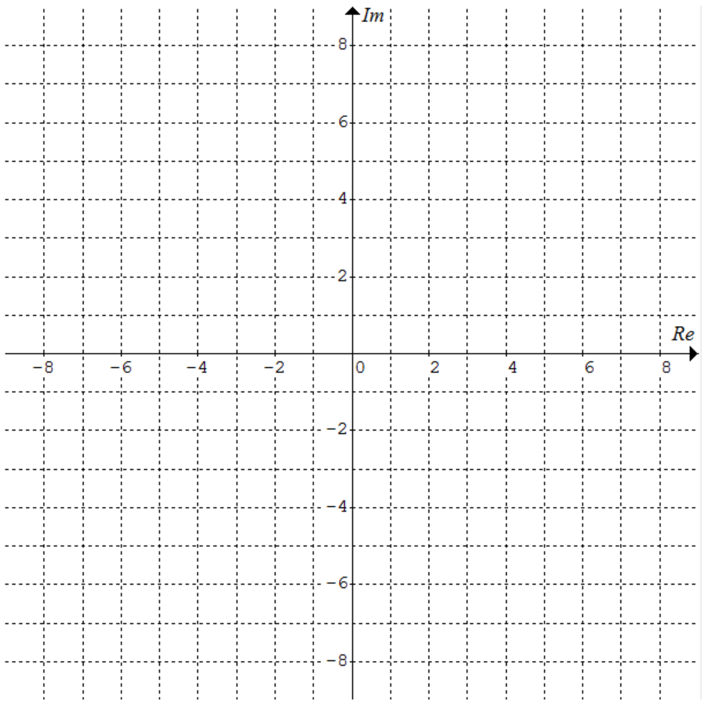
c. Find a unit vector perpendicular to $\tilde{u} + \tilde{v}$ 1 mark

d. Find the vector resolute of $\tilde{v}$ perpendicular to $\tilde{u}$. 2 marks

e. Vector $\tilde{w} = p\,\tilde{i} + q\,\tilde{j} + r\,\tilde{k}$ is perpendicular to both $\tilde{u}$ and $\tilde{v}$. Find the values of p, q and r. 2 marks

Question 2 Consider the set complex numbers given by $\{z : |z+3-4i| = c\}$.

a. Sketch $\{z : |z+3-4i| = c\}$ neatly on the following Argand plane, where $3 < c < 5$. 2 marks



b. In terms of c, find $z \in \{z : |z+3-4i| = c\}$ which is closest to $z = 1+i$. 2 marks

Consider $z = m + ni \in \{z : |z+3-4i| = c\}$ where $c < 5$, a ray from $z = m + ni$ to $z = -3+4i$ and another ray from $z = m + ni$ to $z = 1+i$. The two rays make a right angle.

ci. Show that $(3+m)(1-m) = (4-n)(1-n)$. 2 marks cii. If $c=1$, find $z=m+ni$, correct m and n values to 2 decimal places. 2 marks

ciii. When the value of c varies, show that $z=m+ni\in\left\{z:\left|z+1-\frac{5}{2}i\right|=\frac{5}{2}\right\}$ also. 2 marks

civ. The two rays from $z=m+ni$ to $z=-3+4i$ and $z=1+i$, and the line through $z=-3+4i$ and $z=1+i$ define a triangular region.
Find the maximum area of the triangular region when the value of c varies. 1 mark

Question 3 A particle has position vector $\underset{\sim}{r} = \cot^{-1} t\,\underset{\sim}{i} + \tan^{-1} t\,\underset{\sim}{j}$ where $t \geq 0$ and $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the positive x and y directions respectively.

a. Determine the initial and final positions of the particle. 2 marks

b. Find the Cartesian equation of the particle's path. 1 mark

c. Find the total distance travelled by the particle. 1 mark

d. Write down a unit vector in the direction of the particle's motion. 1 mark

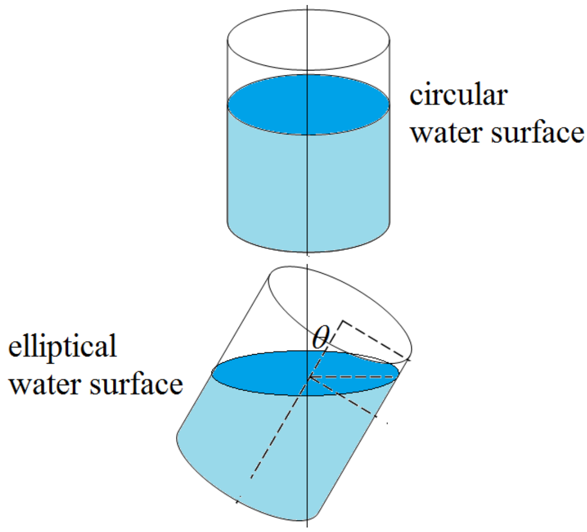
e. Find the speed of the particle when it is half-way through its journey. 1 mark

f. Find the magnitude of the particle's acceleration in terms of t. 2 marks

g. Determine the time when the net force on the particle is a maximum. 1 mark

Question 4 An upright cylindrical water tank has a radius of $\frac{\sqrt{3}}{2}$ m and a height of 2 m. It is filled to a depth of 1.5 m. The tank is tilted at a constant rate, $\frac{d\theta}{dt} = \frac{\pi}{36}$ per second from its upright position.

θ is the tilting angle as shown below.



a. In terms of θ find the area of the elliptical water surface, given area of an ellipse $A = \pi ab$ where a and b are the lengths of the semimajor and semiminor axes.

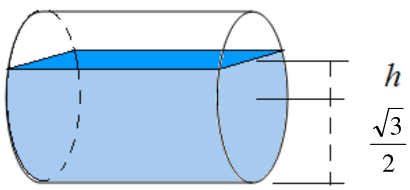
1 mark

b. Determine the rate of change of the water surface area. 2 marks

c. The rate of change of the water surface area in part b is valid in the time interval $[0, T]$. Find T.

1 mark

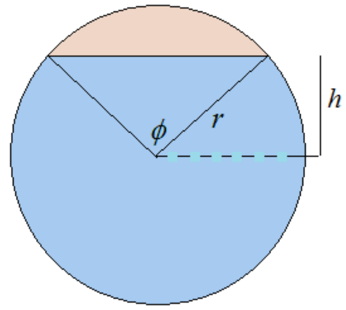
The tilting continues until the cylindrical tank is horizontal, i.e. when $\theta = \frac{\pi}{2}$.



Note: h (m) is measured from the central axis of the cylindrical tank.

d. Show that the uniform cross-sectional area of the water inside the horizontal tank (see diagram below) is $\frac{3\pi}{4}-\frac{3}{4}\cos^{-1}\left(\frac{2h}{\sqrt{3}}\right)+h\sqrt{\frac{3}{4}-h^2}$. 3 marks

Hint: Segment area $=\frac{1}{2}r^2(\phi-\sin\phi)$



e. Find the exact cross-sectional area of the water. 1 mark

Water is pumped out of the horizontal tank at $\frac{\pi}{16}$ m^3 per minute.

f. How long does it take to empty the tank? 1 mark

g. Find the minimum rate of change of the depth of water in the horizontal tank, in cm per minute. Correct your answer to 2 decimal places. 3 marks

h. Find the rate of change of the depth of water when the horizontal tank is a quarter full, in cm per minute. Correct your answer to 2 decimal places. 2 marks

Question 5 A caravan is towed by a car down a slope inclined at $20°$ to the horizontal. The acceleration-time graph from $t = 0$ to $t = 10$ s is shown below. The car comes to a stop at $t = 10$ s.

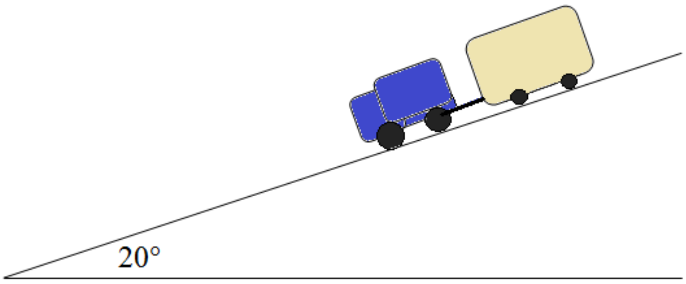
Data for the car:

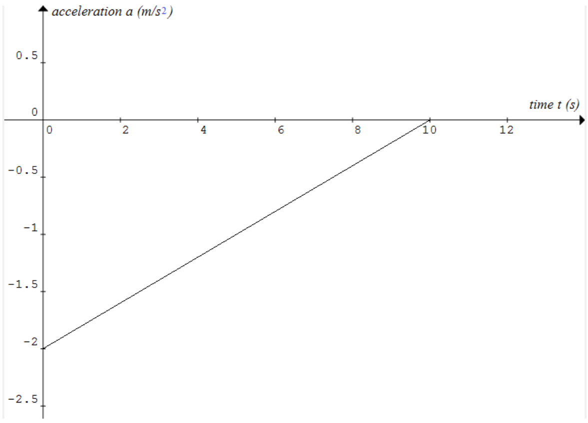
Mass $m_{car} = 1500$ kg, braking force $F_{brake} = ?$, resistive force $F_{resist} = 300$ N, towbar force $F_{tow} = ?$

Data for the caravan:

Mass $m_{caravan} = 1000$ kg, braking force $F_{brake} = 0$, resistive force $F_{resist} = 700$ N, towbar force $F_{tow} = ?$

Assume constant braking and resistive forces while in motion. Express all answers to the nearest whole number.





a. Determine the speed of the car at $t = 0$. 1 mark

b. Calculate the distance travelled by the car from $t = 0$ to $t = 10$. 2 marks

c. On the above diagram draw labeled arrows to show the forces on the car and on the caravan. 2 marks

d. Calculate the braking force F_{brake} provided by the car at $t = 5$ s. 2 marks

e. Calculate the towbar force F_{tow} and its direction on the caravan at $t = 5$ s. 1 mark

Question 6 The weight X kg of a person in a large population of over 18 years old has a normal distribution. Five years ago, the mean and standard deviation of the distribution were 78.2 kg and 4 kg respectively. Assume that the standard deviation remains constant.

a. A random sample of 45 people over 18 years old has a mean weight of 79 kg. 5 more people over 18 years old are randomly selected and combined with the 45 people to form a larger sample.
The weights in kg of the 5 people are 72, 80, 72, 78 and 93.
Determine the value of $\overline{X}$ of the 50 people in the larger random sample. 2 marks

b.. Determine the 95% confidence level for the population μ of X, correct to 2 decimal places. 2 marks

c. The Health Department is concerned whether the weight of the 18 year old population has changed.
Write down suitable hypotheses H_0 and H_1 to test the concern of the Health Department. 2 marks

d. Base on the larger random sample given in part a, state with a reason whether H_0 should be rejected or not rejected at the 5% level of significance.

1 mark

e. A much larger random sample of 400 people over 18 years old is taken and the mean weight is 79 kg.
Base on this much larger random sample, state with a reason whether H_0 should be rejected or not rejected at the 5% level of significance.

1 mark

f. Using the conclusion in part e, does the result in part d indicate a type I error, type II error or no error?
Explain your answer. 2 marks

End of Exam 2

