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2019 Specialist Mathematics Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	E	D	A	B	C	D	A	E	B

11	12	13	14	15	16	17	18	19	20
A	D	C	B	C	A	C	A	B	D

Q1 All graphs of $y = f(|x|)$ are symmetrical about the y-axis. E

Q2 E

Q3 D

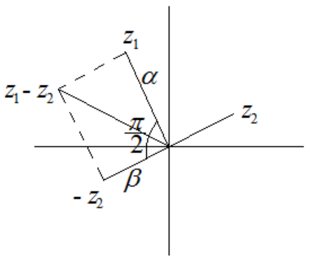
Q4 $y = 2\sin^{-1}\left(\frac{x}{a}-1\right)-b \to y = 2\sin^{-1}\left(\frac{x}{a}\right)-b$

$\to y = -2\sin^{-1}\left(\frac{x}{a}\right)+b$ A

Q5 $z^4+3z^2+4 = z^4+4z^2+4-z^2$

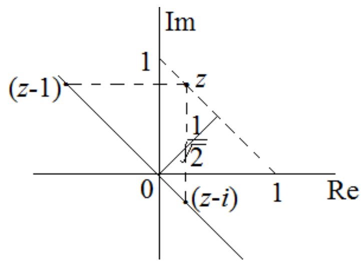
$=(z^2+2)^2-z^2 = (z^2-z+2)(z^2+z+2)$ B

Q6 $\frac{2\pi}{3}-\frac{\pi}{6}=\frac{\pi}{2}$ C



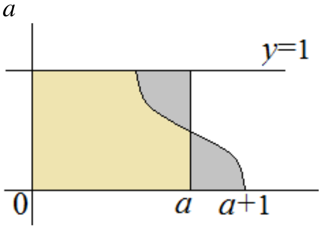
Q7 D

Q8 For $\arg\left(\frac{z-1}{z-i}\right)=\pi$, i.e. $\arg(z-1)-\arg(z-i)=\pi$ to be true, $(z-1)$ and $(z-i)$ must be on each side of the origin opposite to each other. .: z must be a complex number on the dotted line shown below. A



Q9 $x^2+y^2=1$, $\frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}$, $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{1}{\sqrt{1-x^2}}$ E

Q10 Area $= a\times 1 = a$ B



Q11 $\left[\sin^{-1}\left(\frac{x}{\sqrt{a}}\right)\right]_{-\sqrt{a}}^{\sqrt{a}} = \sin^{-1}(1)-\sin^{-1}(-1)=\pi$ A

Q12 $\frac{d^2y}{dx^2}=0$, $x\approx -29.989$, -0.583, 0.572 D

Q13 $m(2\tilde{i}-\tilde{j})+n(\tilde{i}-2\tilde{j}+\tilde{k})+(a\tilde{i}+b\tilde{j}+2\tilde{k})=0$, where $m,n\neq 0$ if the three vectors are linearly dependent.

.: $n+2=0$, $2m+n+a=0$ and $-m-2n+b=0$

.: $a+2b+6=0$ C

Q14 $\tilde{j}.(a\tilde{i}+b\tilde{j}+\tilde{k})=\sqrt{a^2+b^2+1}\cos 120°$

$b=\sqrt{a^2+b^2+1}\left(-\frac{1}{2}\right)$, .: b has a negative value.

.: $b=-\sqrt{\frac{1+a^2}{3}}$ B

Q15 $x=\frac{2t}{t^2+1}$, $y=\frac{t^2-1}{t^2+1}$ and $t\geq 0$

$(0,-1)$ when $t=0$ and $x\geq 0$ for $t\geq 0$

$x^2+y^2=\frac{4t^2+t^4-2t^2+1}{(t^2+1)^2}=1$ C

Q16 Let m kg be the inertial mass. $5.4-9.8m=m\times 1$

$m=0.50$ A

Q17 $a=\frac{F}{m}=\frac{100\cos 60°-30}{m}=\frac{20}{m}$

$\Delta p=m(v-u)=mat=20\times 5=100$ C

Q18 Weight of a bag of apples is independent of the weight of another bag of apples or oranges

$W=X_1+X_2+Y$, $E(W)=E(X_1)+E(X_2)+E(Y)=4.30$

$Var(W)=Var(X_1)+Var(X_2)+Var(Y)=0.0057$,

$sd(W)=\sqrt{0.0057}\approx 0.0755$

$Pr(W<4.00)=normalcdf(-\infty, 4.00, 4.30, 0.0775)\approx 0.000035$ A

Q19 $\bar{X}$: $\mu_{\bar{X}}=E(X)=1.10$, $sd(\bar{X})=\frac{\sigma}{\sqrt{n}}=\frac{\sqrt{0.0016}}{\sqrt{n}}=\frac{0.04}{\sqrt{n}}$

$Pr(\bar{X}<1.092)\approx 0.025$, $Pr\left(Z<\frac{1.092-1.10}{\frac{0.04}{\sqrt{n}}}\right)\approx 0.025$

$\frac{1.092-1.10}{\frac{0.04}{\sqrt{n}}}\approx -1.96$, $n\approx 96$ B

Q20 95% interval: Height $(174.73, 177.27)$, weight $(69.73, 72.27)$

$175\in(174.73, 177.27)$ and $70\in(69.73, 72.27)$

.: no significant evidence to suggest D

SECTION B

Q1

a. $\tilde{u} = \cos 45°\tilde{i} + \cos 60°\tilde{j} + \cos 120°\tilde{k} = \frac{1}{2}\left(\sqrt{2}\tilde{i} + \tilde{j} - \tilde{k}\right)$

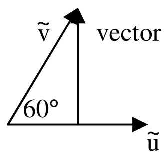
b. Let $\tilde{v} = x\tilde{i} + y\tilde{j} + y\tilde{k}$, $x^2 + y^2 + y^2 = 1$,

$\tilde{v}.\tilde{u} = \cos 60°$, $\therefore \sqrt{2}x + y - y = 1$, $x = \frac{1}{\sqrt{2}}$, $\therefore 2y^2 = \frac{1}{2}$

$\therefore y = \pm\frac{1}{2}$, $\therefore \tilde{v} = \frac{1}{2}\left(\sqrt{2}\tilde{i} - \tilde{j} - \tilde{k}\right)$ or $\frac{1}{2}\left(\sqrt{2}\tilde{i} + \tilde{j} + \tilde{k}\right)$

c. $\tilde{u} + \tilde{v} = \sqrt{2}\tilde{i} + \tilde{j}$, $\tilde{k}$ is a unit vector perpendicular to $\tilde{u} + \tilde{v}$.

d.



vector resolute of $\tilde{v}$ perpendicular to $\tilde{u}$

$= \tilde{v} - \frac{1}{2}\tilde{u} = \frac{1}{4}\left(\sqrt{2}\tilde{i} + \tilde{j} + 3\tilde{k}\right)$

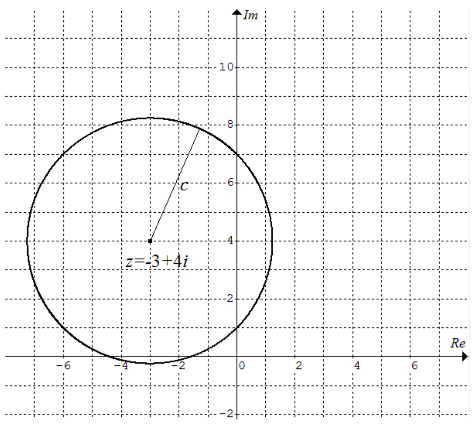
e. $\tilde{w}.\tilde{u} = 0$ and $\tilde{w}.\tilde{v} = 0$

$\therefore \frac{1}{2}\left(\sqrt{2}p + q - r\right) = 0$ and $\frac{1}{2}\left(\sqrt{2}p + q + r\right) = 0$

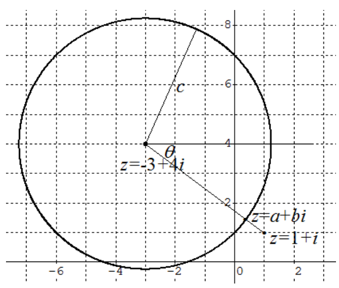
$\therefore r = 0$ and $\sqrt{2}p + q = 0$ where $p \in R \setminus \{0\}$ and $q = -\sqrt{2}p$

Q2

a.



b. Let the required complex number be $z = a + bi$.



$\text{Arg}(a + bi + 3 - 4i) = \text{Arg}(1 + i + 3 - 4i)$,

$\text{Arg}(a + 3 + (b - 4)i) = \text{Arg}(4 - 3i)$,

$\therefore \tan^{-1}\left(\frac{b-4}{a+3}\right) = \tan^{-1}\left(\frac{-3}{4}\right) = \theta$, $\therefore \cos\theta = \frac{4}{5}$, $\sin\theta = \frac{-3}{5}$

$a = -3 + c\cos\theta$ and $b = 4 + c\sin\theta$

$\therefore z = \left(-3 + \frac{4c}{5}\right) + \left(4 - \frac{3c}{5}\right)i$

ci. $\text{Arg}(-3 + 4i - m - ni) - \text{Arg}(1 + i - m - ni) = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

i.e. $\tan^{-1}\left(\frac{4-n}{-3-m}\right) - \tan^{-1}\left(\frac{1-n}{1-m}\right) = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

$\therefore \tan\left(\tan^{-1}\left(\frac{4-n}{-3-m}\right) - \tan^{-1}\left(\frac{1-n}{1-m}\right)\right)$ is undefined

$\therefore \frac{\frac{4-n}{-3-m} - \frac{1-n}{1-m}}{1 + \left(\frac{4-n}{-3-m}\right)\left(\frac{1-n}{1-m}\right)}$ is undefined $\therefore 1 + \left(\frac{4-n}{-3-m}\right)\left(\frac{1-n}{1-m}\right) = 0$

$\therefore (3 + m)(1 - m) = (4 - n)(1 - n)$

cii. $c = 1$, $|(m + ni) + (3 - 4i)| = 1$, $\therefore (3 + m)^2 + (4 - n)^2 = 1$

Solve $(3 + m)(1 - m) = (4 - n)(1 - n)$ and $(3 + m)^2 + (4 - n)^2 = 1$

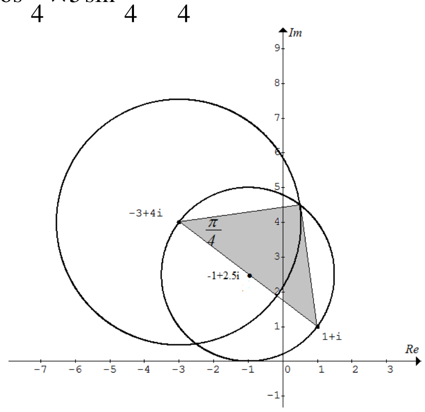
$m \approx -3.43$ and $n \approx 3.01$ or $m \approx -2.25$ and $n \approx 4.66$

ciii. $(3 + m)(1 - m) = (4 - n)(1 - n)$, $\therefore 3 - 2m - m^2 = 4 - 5n + n^2$

$\therefore (m + 1)^2 + \left(n - \frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2$, $\therefore z = m + ni \in \left\{z : \left|z + 1 - \frac{5}{2}i\right| = \frac{5}{2}\right\}$

civ. The shaded triangle in the diagram below is the largest possible when the value of c varies and the angle is at $\frac{\pi}{4}$.

$\text{Area} = \frac{1}{2} \times 5\cos\frac{\pi}{4} \times 5\sin\frac{\pi}{4} = \frac{25}{4}$



Q3a. $\tilde{r}(t) = \left(\frac{\pi}{2} - \tan^{-1} t\right)\tilde{i} + \tan^{-1} t\,\tilde{j}$

Initial: $t = 0$, $\tilde{r} = \frac{\pi}{2}\tilde{i}$, final: $t \to \infty$, $\tilde{r} \to \frac{\pi}{2}\tilde{j}$

b. $x = \frac{\pi}{2} - \tan^{-1} t$, $y = \tan^{-1} t$, $\therefore x + y = \frac{\pi}{2}$ where $x \in \left(0, \frac{\pi}{2}\right]$

c. Total distance $\to \sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^2} = \frac{\pi}{\sqrt{2}}$

d. $-\frac{1}{\sqrt{2}}\tilde{i} + \frac{1}{\sqrt{2}}\tilde{j}$

e. $\tilde{v} = -\frac{1}{1+t^2}\tilde{i} + \frac{1}{1+t^2}\tilde{j}$, half-way: $y = \tan^{-1} t = \frac{\pi}{4}$, $t = 1$ and

$\tilde{v} = -\frac{1}{2}\tilde{i} + \frac{1}{2}\tilde{j}$, speed $= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$

f. $\tilde{a} = \frac{2t}{(1+t^2)^2}\tilde{i} - \frac{2t}{(1+t^2)^2}\tilde{j}$, $a = |\tilde{a}| = \frac{2\sqrt{2}t}{(1+t^2)^2}$

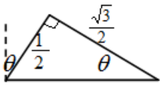
g. Maximum acceleration when $\frac{da}{dt} = 0$, $t = \frac{1}{\sqrt{3}}$

.: net force is maximum at $t = \frac{1}{\sqrt{3}}$.

Q4a. $A = \pi\left(\frac{\frac{\sqrt{3}}{2}}{\cos\theta}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3\pi}{4\cos\theta}$

b. $\frac{dA}{dt} = \frac{dA}{d\theta}\times\frac{d\theta}{dt} = \frac{\pi^2}{48}\sec\theta\tan\theta$ m^{2}s^{-1}

c. $\tan\theta = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$, $\theta = \frac{\pi}{6}$, $T = \frac{\frac{\pi}{6}}{\frac{\pi}{36}} = 6$ s



d. $\cos\frac{\phi}{2} = \frac{h}{r} = \frac{2h}{\sqrt{3}}$, $\phi = 2\cos^{-1}\left(\frac{2h}{\sqrt{3}}\right)$

$\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2} = 2\left(\frac{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - h^2}}{\frac{\sqrt{3}}{2}}\right)\left(\frac{2h}{\sqrt{3}}\right) = \frac{8h\sqrt{\frac{3}{4}-h^2}}{3}$

Segment area

$= 2\left(\frac{\sqrt{3}}{2}\right)^2\left(2\cos^{-1}\left(\frac{2h}{\sqrt{3}}\right) - \frac{8h\sqrt{\frac{3}{4}-h^2}}{3}\right) = \frac{3}{4}\cos^{-1}\left(\frac{2h}{\sqrt{3}}\right) - h\sqrt{\frac{3}{4}-h^2}$

Cross-sectional area $A_c = \left(\frac{\sqrt{3}}{2}\right)^2\pi - \frac{3}{4}\cos^{-1}\left(\frac{2h}{\sqrt{3}}\right) - h\sqrt{\frac{3}{4}-h^2}$

$= \frac{3\pi}{4} - \frac{3}{4}\cos^{-1}\left(\frac{2h}{\sqrt{3}}\right) - h\sqrt{\frac{3}{4}-h^2}$

e. Volume of water $= \pi\left(\frac{\sqrt{3}}{2}\right)^2 \times 1.5 = 2\times A_c$, .: $A_c = \frac{9\pi}{16}$ m^2

f. Time required to empty $= \frac{\frac{9\pi}{16}\times 2}{\frac{\pi}{16}} = 18$ min

g. At depth h, $V = 2\left(\frac{3\pi}{4} - \frac{3}{4}\cos^{-1}\left(\frac{2h}{\sqrt{3}}\right) - h\sqrt{\frac{3}{4}-h^2}\right)$

Minimum rate of change of the depth $\frac{dh}{dt}$ when the tank is half full, i.e. when $h = 0$

.: $\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{\frac{\pi}{16}}{\frac{dV}{dh}} \approx 0.0567$ m min^{-1} by CAS or 5.67 cm min^{-1}

h. $\frac{dh}{dt}$ has the same value at $\frac{1}{4}$ full and $\frac{3}{4}$ full.

At the start the tank is $\frac{3}{4}$ full and $A_c = \frac{9\pi}{16}$, by CAS $h \approx 0.34985$.

.: $\frac{dh}{dt} = \frac{\frac{\pi}{16}}{\frac{dV}{dh}} \approx 0.0620$ m min^{-1} by CAS or 6.20 cm min^{-1}.

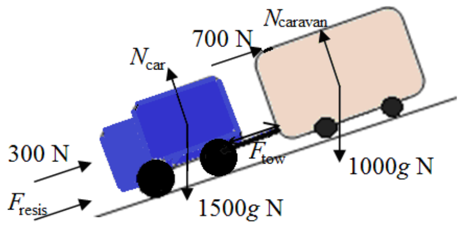
Q5a. $\Delta v = \text{area} = \frac{1}{2}(-2)(10) = -10$ m s^{-1}.

At $t = 0$, speed $= 10$ m s^{-1}.

b. $a = -0.2t$, $v = -0.1t^2 + 10$, $s = -\frac{0.1t^3}{3} + 10t$

At $t = 10$, $s = \frac{200}{3}$ m, distance $= \frac{200}{3}$ m

c. Given $F_{\text{brake}} = 0$ for caravan.



d. $t = 5$, $a = -1.0$

Consider the car and the caravan as one object, $F_{\text{net}} = ma$

$F_{\text{brake}} + {}^{-}300 + {}^{-}700 + 2500g\sin 20° = 2500\times{}^{-}1.0$, $F_{\text{brake}} \approx -9880$ N

e. Consider the caravan only,

$F_{\text{tow}} + 1000g\sin 20° + {}^{-}700 = 1000\times{}^{-}1.0$

$F_{\text{tow}} \approx {}^{-}3652$ N, i.e. 3652 N up the slope

Q6a. $\bar{x} = \frac{45\times 79 + 72 + 80 + 72 + 78 + 93}{50} = 79$

b. $\left(79 - 1.96\times\frac{4}{\sqrt{50}},\ 79 + 1.96\times\frac{4}{\sqrt{50}}\right) \approx (77.89, 80.11)$

c. H_0: Has not changed, $\mu = 78.2$; H_1: Has changed, $\mu \neq 78.2$

d. Since $78.2 \in (77.89, 80.11)$, H_0 should not be rejected at the 5% level of significance.

e. The 95% confidence level for the much larger random sample of 400 people is

$\left(79 - 1.96\times\frac{4}{\sqrt{400}},\ 79 + 1.96\times\frac{4}{\sqrt{400}}\right) \approx (78.61, 79.39)$

Since $78.2 \notin (78.61, 79.39)$, H_0 should be rejected at the 5% level of significance.

f. The result in part d indicates a type II error. Larger sample size should be used to give a more accurate result, and the conclusion in part e was correct.

In part d the conclusion was not correct. H_0 was in fact not true as shown in part e, but it was not rejected.

Please inform mathline@itute.com re conceptual and mathematical errors