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# 2019 Specialist Mathematics

## Year 12 Application Task

**(Time allowed: 4 hours plus)**

## Modeling Task/ Problem Solving Task

**(Time allowed: 2 hours plus)**

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# Theme: Isomorphism in 2-dimensional vectors and Complex numbers

***Application Task (Part I and Part II)***
***Problem Solving Task (Part I only)***
***Modeling Task (Part II only)***

Two-dimensional vectors and complex numbers are isomorphic in some respects, meaning they are the same in form and properties in those respects. They can both be used to define or specify the locus of a point in either the Cartesian plane or the Argand plane.

# Part I

**Question 1**

**a.** The magnitude of vector $2\tilde{i} - \sqrt{3}\tilde{j}$ in the Cartesian plane has the same value as the modulus of complex number $2 - i\sqrt{3}$ in the Argand plane. Show that the statement is true.

**b.** Find the angle (radians) made by unit vector $\tilde{i}$ and $2\tilde{i} - \sqrt{3}\tilde{j}$, and the Argument of $2 - i\sqrt{3}$.

**c.** Write down the results when $2\tilde{i} - \sqrt{3}\tilde{j}$ and $2 - i\sqrt{3}$ are multiplied by real number 2.
Give a geometric interpretation in each case.

**d.** Summarise the isomorphic results found in parts **a**, **b** and **c** for any vector $x\tilde{i} + y\tilde{j}$ and any complex number $x + iy$ where $x, y \in R$, and real number $k$.

| | Vector $x\tilde{i} + y\tilde{j}$ | Complex number $x + iy$ |
|---|---|---|
| Magnitude/modulus $r$ | | |
| Angle/argument $\theta$ | | |
| Multiply by $k$ | | |



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**e.** Write down the result when $2 - i\sqrt{3}$ is multiplied by $i$. Give a geometric interpretation of this operation.

**f.** What is the operation to be carried out on $2\tilde{i} - \sqrt{3}\tilde{j}$ to produce the same geometric result as in part **e**?
Hint: Write $2\tilde{i} - \sqrt{3}\tilde{j}$ as a column matrix. Find a $2 \times 2$ matrix to multiply the column matrix, giving the desired result.

**g.** Express each of $2\tilde{i} - \sqrt{3}\tilde{j}$ and $2 - i\sqrt{3}$ in polar form.

**h.** Write down the result when $2 - i\sqrt{3}$ in polar form is multiplied by $\text{cis}\left(\frac{\pi}{3}\right)$.

Give a geometric interpretation of this operation.


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**i.** Determine the operation to be carried out on $2\tilde{i} - \sqrt{3}\tilde{j}$ to produce the same geometric result as in part **h**.

**j.** Write down the inverse operation for each of parts **h** and **i**.

**k.** For part **h**, write down a general (i.e. in terms of argument $\theta$) statement of the operation on complex number $x + iy$ and its geometric interpretation.

**l.** For part **i** write down a general (i.e. in terms of angle $\theta$) statement of the operation on vector $x\tilde{i} + y\tilde{j}$ and its geometric interpretation.


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Vectors and complex numbers are isomorphic with respect to the specified operation in parts **k** and **l**. The following will investigate isomorphism of vectors and complex numbers with respect to addition or subtraction.

Consider free vectors $\tilde{p} = 2\tilde{i} - \sqrt{3}\tilde{j}$ and $\tilde{q} = \tilde{i} + \tilde{j}$.

**m.** Find the magnitude and direction (angle with $\tilde{i}$) of $\tilde{p} + \tilde{q}$.

**n.** Write down the complex numbers (labeled as $p$ and $q$) corresponding to $\tilde{p}$ and $\tilde{q}$ respectively. Find the modulus and Argument of $p + q$.

Consider another two free vectors $\tilde{a} = \tilde{i} + \sqrt{3}\tilde{j}$ and $\tilde{b} = \sqrt{3}\tilde{i} + \tilde{j}$. You may consider the corresponding complex numbers $a$ and $b$ to answer the following questions.

**o.** Show that the direction (angle with $\tilde{i}$) of $\tilde{a} + \tilde{b}$ is an exact value.

**p.** Find a different pair of free vectors such that the direction (angle with $\tilde{i}$) of their sum is an exact value.


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Consider two general free vectors $\tilde{c} = s\tilde{i} + u\tilde{j}$ and $\tilde{d} = v\tilde{i} + w\tilde{j}$.
Let $\phi$ be the direction (angle with $\tilde{i}$) of $\tilde{c} + \tilde{d}$.
You may work with corresponding complex numbers and arguments.

**q.** Find $\tan\phi$ in terms of $s$, $u$, $v$ and $w$.

When $s + v = 2$ and $u + w = 2\sqrt{3}$, e.g. $s = 0.9$, $v = 1.1$, $u = 0.7\sqrt{3}$ and $w = 1.3\sqrt{3}$, then $\tilde{c} = 0.9\tilde{i} + 0.7\sqrt{3}\tilde{j}$, $\tilde{d} = 1.1\tilde{i} + 1.3\sqrt{3}\tilde{j}$, and $\phi$ has an exact value of $\frac{\pi}{6}$.

**r.** Find a possible pair of free vectors $\tilde{c} = s\tilde{i} + u\tilde{j}$ and $\tilde{d} = v\tilde{i} + w\tilde{j}$ such that $\phi = \frac{2\pi}{3}$.

**s.** Find a possible pair of free vectors $\tilde{c} = s\tilde{i} + u\tilde{j}$ and $\tilde{d} = v\tilde{i} + w\tilde{j}$ such that $\phi = -\frac{3\pi}{4}$.


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Let $r$ be the magnitude of $\tilde{c} + \tilde{d}$.
You may work with corresponding complex numbers and moduli.

**t.** Find $r$ in terms of $s$, $u$, $v$ and $w$.

**u.** Find a possible pair of free vectors $\tilde{c} = s\tilde{i} + u\tilde{j}$ and $\tilde{d} = v\tilde{i} + w\tilde{j}$ such that $r$ is an integer.

**v.** Is it possible to have a pair of free vectors $\tilde{c} = s\tilde{i} + u\tilde{j}$ and $\tilde{d} = v\tilde{i} + w\tilde{j}$ such that $r$ is an integer and $\tan\phi$ is an exact value? If your answer is yes, find the pair of free vectors.

**w.** With respect to certain operation isomorphism of vectors and complex numbers is/is not true. Find either an operation that isomorphism is true, or an operation that isomorphism is not true. Choose one only and explain your answer.


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# Part II

The locus/path of a point can be specified by a two-dimensional position vector $\tilde{r}(t) = x(t)\tilde{i} + y(t)\tilde{j}$ on the Cartesian plane, or a complex number $z(t) = x(t) + i\,y(t)$ on the Argand plane, where $x$ and $y$ are functions of parameter $t$ and $t \geq 0$.

Consider the locus of point *P* given by
$\tilde{r}_P(t) = \sqrt{2}\cos(t)\tilde{i} + \sqrt{2}\sin(t)\tilde{j}$ or $z_P(t) = \sqrt{2}\cos(t) + i\sqrt{2}\sin(t)$.

**a.** Determine the Cartesian equation of the locus of *P* to show that it is a circle.
State its centre and radius.
Determine the starting position (or complex number) when $t = 0$.
Calculate the distance travelled by *P* in the interval $t \in \left[\frac{\pi}{4}, \frac{10\pi}{4}\right]$.



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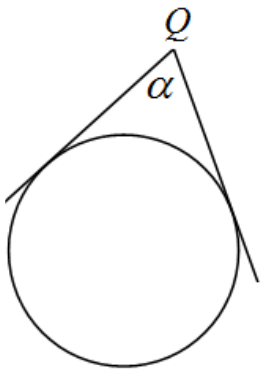
Two tangent lines of the circle intersect at point $Q$. The two lines make a constant angle of $\alpha$.

The locus of $Q$ is given by

$$\tilde{r}_Q(t) = \sqrt{2}\text{cosec}\left(\frac{\alpha}{2}\right)\cos(t)\tilde{i} + \sqrt{2}\text{cosec}\left(\frac{\alpha}{2}\right)\sin(t)\tilde{j}\,.$$




**b.** Find the Cartesian equation of the locus of $Q$ to show that it is a circle.
State its centre and radius in terms of $\alpha$.
Show that points $O$ (the origin), $P$ and $Q$ are always collinear.
Calculate the distance (in terms of $\alpha$) travelled by $Q$ in the interval $t \in \left[\frac{\pi}{4}, \frac{10\pi}{4}\right]$.

The two circles in parts **a** and **b** are concentric, i.e. they have the same centre.
The region between any two concentric circles is called the annulus.

**c.** Show that $\dfrac{\text{area of inner circle}}{\text{area of annulus}} = \dfrac{1-\cos(\alpha)}{1+\cos(\alpha)}$.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Consider the locus of point *R* in the Argand plane given by $z_R(t) = t\cos(t) + it\sin(t)$ where $t \geq 0$.

**d.** Choose twelve or more suitable values of $t \in [0, 5\pi]$ in ascending order to set up a table of values for $\text{mod}(z_R)$ and $\arg(z_R)$.

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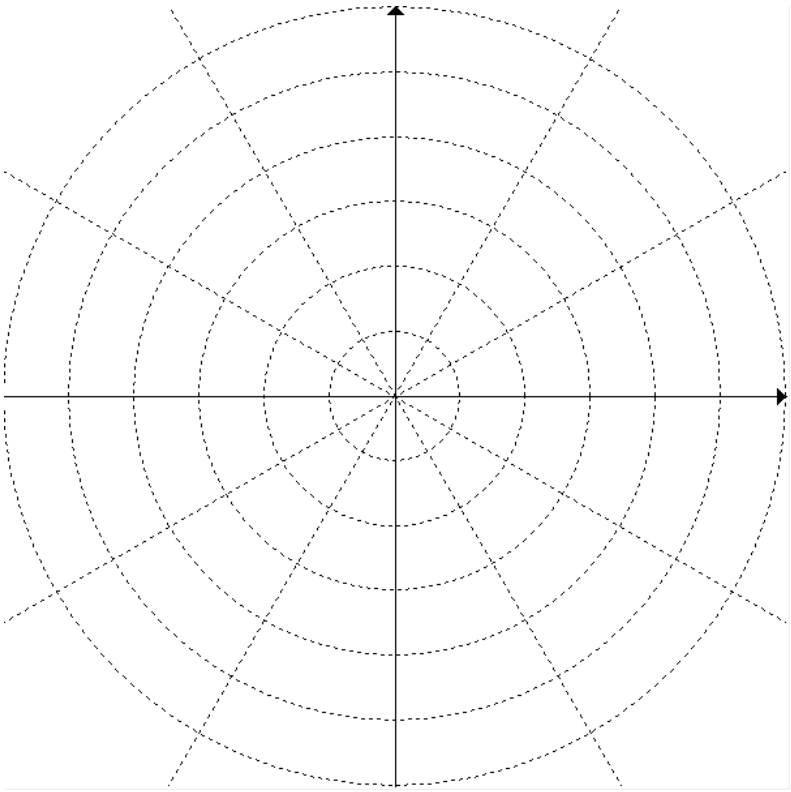
**e.** Plot the complex numbers and draw a smooth curve to show the locus of point *R*.



**f.** Determine the distance travelled by point *R* in the interval $t \in [10, 12.5]$, correct to 1 decimal place.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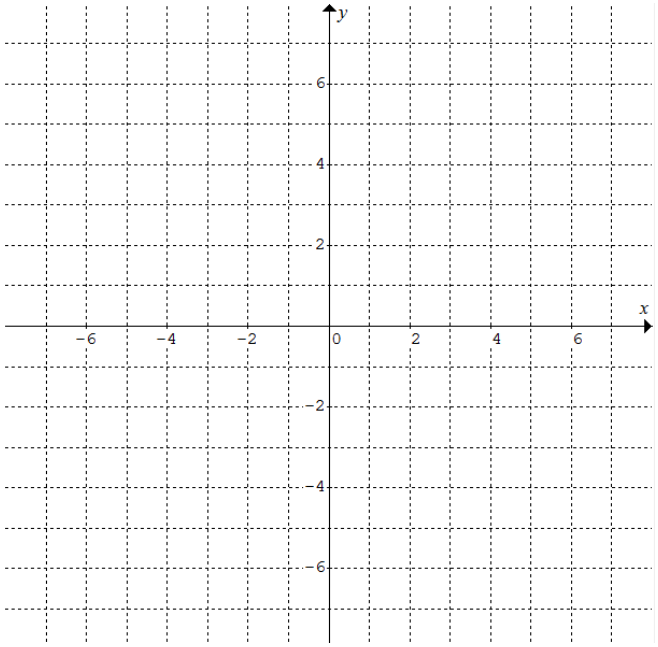
Consider the locus of point *S* in the Cartesian plane given by
$\tilde{r}_S(t) = 2a(\cos(t)+1)\tilde{i} + a(\sin(t)-1)\tilde{j}$ where $t \geq 0$ and $a \in R^+$.

**g.** Let $a = 1$. Determine the Cartesian equation of the locus of point *S*.

**h.** Sketch the locus of point *S* for $a = 1$.




**i.** Choose two other suitable values of $a > 1$.
Determine the equation and sketch each locus of point *S* on the same axes as shown above.
Label each locus with its *a* value. Choose a different *a* value if the locus is outside the grids.

**j.** Discuss the effects of changing *a* on the shape, size and location of the locus of point *S*.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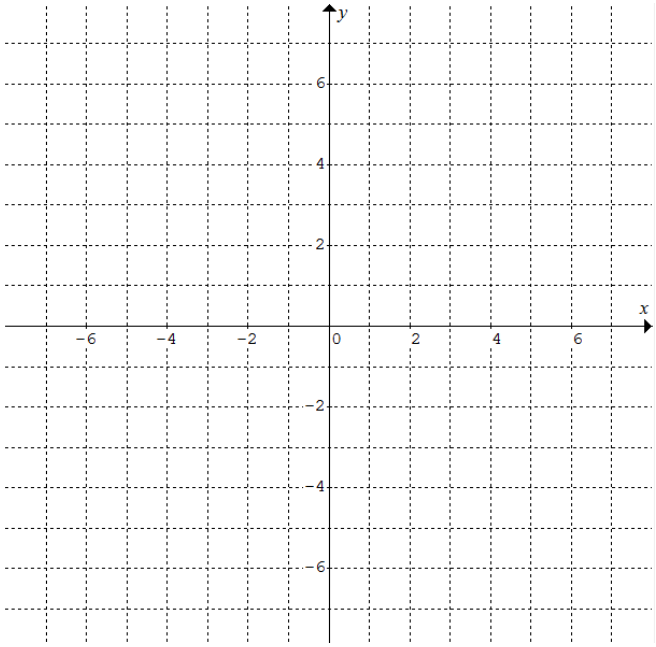
Consider the locus of point $T$ in the Argand plane given by

$z_T(t) = \left(1 + \tan\frac{(t+1)\pi}{3}\tan\frac{t\pi}{3}\right) + i\left(\tan\frac{(t+1)\pi}{3} - \tan\frac{t\pi}{3}\right)$ where $t \geq 0$.

**k.** Simplify $\frac{\text{Im}(z_T)}{\text{Re}(z_T)}$.

**l.** Hence find the locus of point $T$ in the Argand plane in terms of $\text{Re}(z_T)$ and $\text{Im}(z_T)$.

**m.** Sketch accurately the locus of point $T$.




## End of Task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