2019 Specialist Mathematics

Year 12 Modelling Task

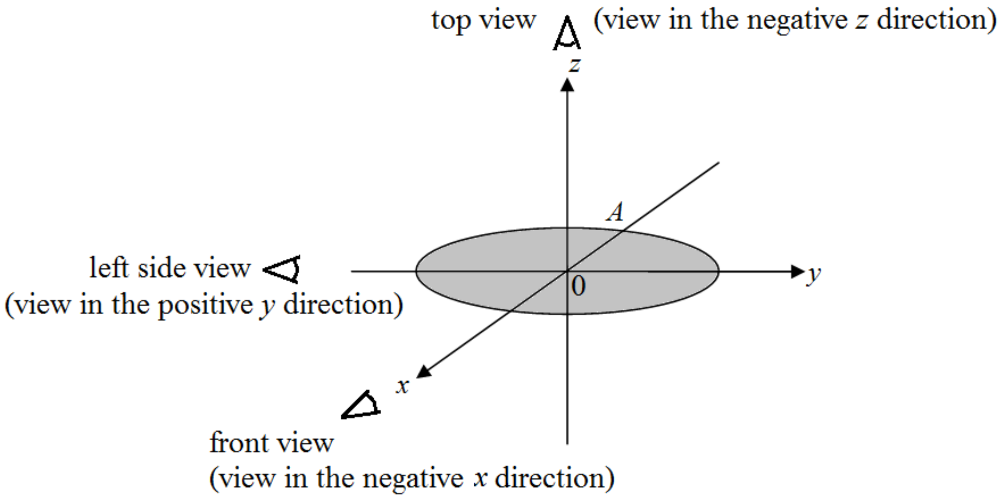
(Time allowed: 2.0 hours plus)

Modelling Task

Theme: Circular and helical motions

Assumed knowledge:
Functions, graphs, vectors, kinematics, parametric equations, vector calculus and use of CAS

Specifications:
x and y axes are on a horizontal plane.
z axis is perpendicular to the horizontal plane and directed upwards.
$\tilde{i}$, $\tilde{j}$ and $\tilde{k}$ are unit vectors in the direction of x, y and z axes respectively.
Time $t \geq 0$ is measured in seconds and distance is in metres.



Part I (60 – 75 minutes)

The top view shows particle P moving in a horizontal circle and completing one revolution in 2 seconds.
The particle is at point A when $t = 0$.
It moves in the anticlockwise direction at a constant speed of 2π m s^{-1}.

a. Show that the radius of the circle is 2 m.

The left side view shows the particle appearing to move along the x axis.

b. The scalar $\tilde{i}$ component of the particle's position is $x(t) = a\cos(bt)$.
Explain why $a = -2$ and $b = \pi$.

The front view shows the particle appearing to move along the y axis.
c. The scalar $\tilde{j}$ component of the particle's position is $y(t) = p\sin(qt)$.
Find the value of each of p and q.

d. Hence find the particle's position vector $\tilde{r}(t)$ at time t.

Another particle, Q, has position vector given by $\tilde{r}(t) = 2\sin(\pi t)\tilde{i} + 2\cos(\pi t)\tilde{j}$.

e. Will the two particles P and Q collide? Explain your answer.
If your answer is yes find the time and place (x and y coordinates) of collision.

A third particle, S, has position vector given by $\tilde{r}(t) = -w\cos\left(\frac{t}{w}\right)\tilde{i} + w\sin\left(\frac{t}{w}\right)\tilde{j}$ where $w \in R^+$.

f. Find the position, $\tilde{r}(0)$, of the third particle in terms of w if necessary.

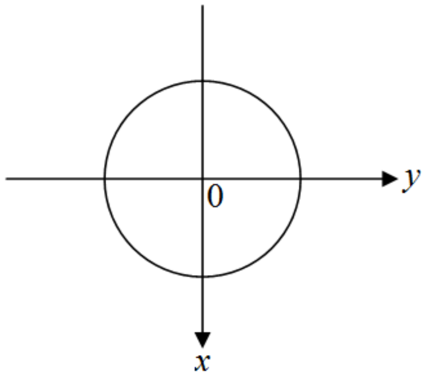
g. Find the velocity, $\tilde{v}(0)$, of the third particle in terms of w if necessary.

h. Find the acceleration, $\tilde{a}(0)$, of the third particle in terms of w if necessary.

i. Show that $\tilde{v}$ is perpendicular to $\tilde{r}$ at time t.

j. Express $\tilde{a}$ in terms of $\tilde{r}$ and w at time t.

k. The following diagram shows the top view of the path of particle S for $w = 2$.
Draw accurately on the diagram the velocity and acceleration vectors at $t = 0$, showing the correct length and direction for each.



Particle T moves in the same plane as the other particles.
Its velocity vector is given by $\tilde{v}(t) = -\pi\sin(\pi t)\tilde{i} + \pi\cos(\pi t)\tilde{j}$, $t \geq 0$

l. Find the initial ($t = 0$) position vector of particle T. Give full and clear explanation.

Particle X moves in the same plane as the other particles.
It is at $(1,0)$ when $t = 0$.
Its velocity vector is given by $\tilde{v}(t) = -2\pi t\sin(\pi t^2)\tilde{i} + 2\pi t\cos(\pi t^2)\tilde{j}$.

m. Show that particle X moves in a circle .

n. Find t when it returns to $(1,0)$ for the first time.

o. Find t when it returns to $(1,0)$ for the second time.

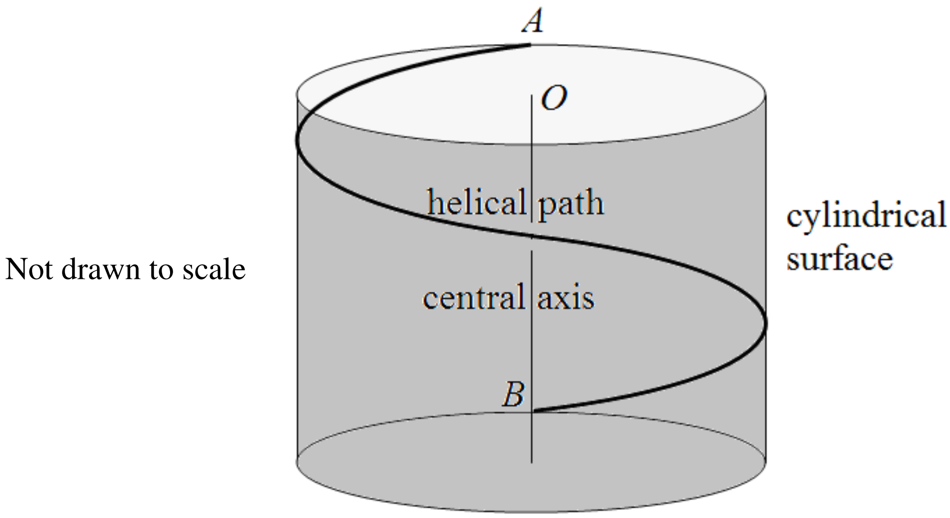
p. In terms of n find the time taken for particle X to complete n revolutions.

q. Find $\lim_{n\to\infty}(t_{n+1} - t_n)$ where t_n is the time taken for particle X to complete n revolutions. Explain.

r. Let θ be the angle between $\tilde{v}(t)$ and $\tilde{a}(t)$. Find θ in radians when (i) $t = \sqrt{2}$ and (ii) $t = 2$.

s. Express $\cos\theta$ in terms of t.
Hence find θ (i) as $t \to 0$, and (ii) as $t \to \infty$. Explain.

Part II (60 – 75 minutes)



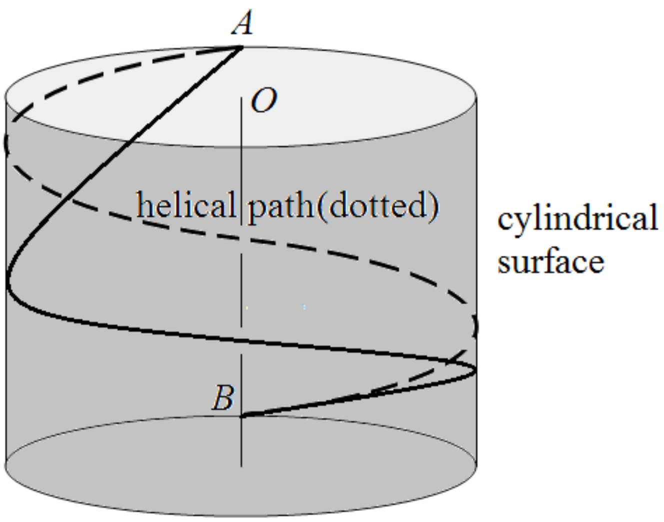
The central axis of the cylindrical surface in the diagram is on the z - axis pointing vertically upwards.
Point O is the origin of the axes.
The y - axis (not shown) is directed from left to right passing through the origin O.
The x - axis (not shown) is directed through point A and then O towards the reader.

Particle W moves from point A to point B along the helical path (solid curve) on the cylindrical surface.

The position vector of particle W is $\tilde{r}(t) = -2\left(\cos(\pi t)\tilde{i} + \sin(\pi t)\tilde{j} + t\tilde{k}\right)$.

a. Show that the vertical distance fallen by particle W is 4 metres.

b. Use the formula $\text{arc length} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ to verify that the length of the helical path is $4\sqrt{\pi^2 + 1}$ metres.



c. If particle W moves from point A to point B along a different path (an example is shown in the above diagram), traversing forward without returning to the same point across the entire cylindrical curved surface, show that $4\sqrt{\pi^2+1} < \ell < 4(\pi+1)$ where ℓ is the length of the path.

If the position vector of particle W is $\tilde{r}(t) = -\alpha\left(\cos(\beta\pi t)\tilde{i} + \sin(\beta\pi t)\tilde{j} + t\tilde{k}\right)$ where $\alpha, \beta \in R^+$,

d. find the length of the helical path from point A to point B in terms of α and β

e. show that the acceleration vector is horizontal at time t

If the position vector of particle W is $\tilde{r}(t) = -2\left(\cos(n\pi t^2)\tilde{i} + \sin(n\pi t^2)\tilde{j} + t^2\tilde{k}\right)$ where $n \in R^+$, and the helical path is the same (in radius and height) as in **part a**,

f. show that the time taken to travel from point A to point B is $\sqrt{2}$.

g. show that $n = 1$

h. calculate the distance travelled by particle W from $t = \sqrt{2}$ to $t = 2$ if the helical path continues beyond point B

i. determine (i) the average velocity and (ii) the average speed in the interval $0 \le t \le \sqrt{2}$, express each answer in simplest surd form with rationalised denominator

j. calculate $\left|\tilde{v}(0)\right|$ and $\left|\tilde{a}(0)\right|$

k. describe the direction of the acceleration vector at time t relative to the horizontal plane and the central axis

End of Task