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2019 VCAA Mathematical Methods Exam 2 Solutions

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SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	B	E	C	C	A	D	C	E	D

11	12	13	14	15	16	17	18	19	20
A	E	C	B	A	A	D	D	E	D

Q1 Period $=\frac{2\pi}{\frac{2}{5}}=5\pi$, range is $[-5,1]$ **B**

Q2 $\Delta=4+4k>0$, $k>-1$ **B**

Q3 $f(6)=\frac{a}{2}$, $f(8)=\frac{a}{4}$, av rate of change $=\frac{\frac{a}{4}-\frac{a}{2}}{4-2}=-\frac{a}{8}$ **E**

Q4 $[-a\cos x+b\sin x]_0^{\frac{\pi}{6}}=\frac{-a\sqrt{3}}{2}+\frac{b}{2}+a=\frac{-a\sqrt{3}+b+2a}{2}$ **C**

Q5 $f(x)=x^3-x^2+c$ and $f(4)=0$, .: $c=-48$ **C**

Q6 $V(x)=x(80-2x)(50-2x)=4x(40-x)(25-x)$
By CAS, $V_{\max}$ occurs at $x=10$ **A**

Q7 $16a=1$, $a=\frac{1}{16}$, $\overline{X}=1\times\frac{3}{16}+2\times\frac{5}{16}+3\times\frac{7}{16}=\frac{17}{8}$ **D**

Q8 $\Pr(X=74 \mid X\ge 70)=\frac{\Pr(X=74)}{\Pr(X\ge 70)}\approx 0.1494$ **C**

Q9 $\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}\\ -2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$, $a=1$ and $b=-1$ **E**

Q10 **D**

Q11 $\Pr(B\mid A)=m$, $\Pr(B\mid A')=n$
If A and B are independent, then $\Pr(B)$ is not conditional on the occurrence of A or not. .: $m=n$ **A**

Q12 $\int_1^2 (f(x)+x)\,dx=\int_1^2 f(x)dx+\int_1^2 x\,dx=4-(-2)+\frac{3}{2}=\frac{15}{2}$ **E**

Q13 $h(x)$ is the result of a horizontal dilation of $f(x)$ by a factor of 2 followed by a vertical translation of 5 units.
$(-2,7)\to(-2\times 2,7+5)=(-4,12)$ **C**

Q14 $\Pr(X>190)=\Pr\left(Z>\frac{190-200}{\sigma}\right)=\Pr\left(Z>\frac{-10}{\sigma}\right)=0.97$ or
$\Pr\left(Z<\frac{-10}{\sigma}\right)=0.03$, $\sigma\approx 5.3$ g **B**

Q15 $f'(x)=2x-4$, $f'(5)=6$, $g'(7)=\frac{1}{f'(5)}=\frac{1}{6}$ **A**

Q16 Gradient is 0 at $x=0,5$. Gradient is negative in $(-\infty,0)$ or $(0,5)$ and positive in $(5,\infty)$. **A**

Q17 $\Pr(\text{two of the same colour})=\frac{{}^kC_2+{}^{n-k}C_2}{{}^nC_2}$
$=\frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}$ **D**

Q18 Average value $=\frac{\text{area}}{a+b}=\frac{1}{a+b}=\frac{3}{4}$, .: $b=\frac{4}{3}-a$
where area $=a^2+\frac{1}{2}(2a+b)b=1$.
Substitute $b=\frac{4}{3}-a$ in the equation and solve for $a^2=\frac{2}{9}$
.: $\Pr(X>0)=1-\Pr(X<0)=1-a^2=\frac{7}{9}$ **D**

Q19 $\tan 2x=d$, $2x=\alpha$, $\pi+\alpha$ or $2\pi+\alpha$
.: $x=\frac{\alpha}{2}$, $\frac{\pi+\alpha}{2}$ or $\frac{2\pi+\alpha}{2}$
.: sum of the solutions $=\frac{3(\pi+\alpha)}{2}$ **E**

Q20 $\log_x y+\log_y z=\frac{\log_y y}{\log_y x}+\frac{\log_z z}{\log_z y}=\frac{1}{\log_y x}+\frac{1}{\log_z y}$ **D**

SECTION B

Q1a $f'(x)=-2x^3e^{-x^2}+2xe^{-x^2}=2xe^{-x^2}(1-x^2)$

Q1bi Local minimum

Q1bii Let $2xe^{-x^2}(1-x^2)=0$, $x=-1,0,1$ where stationary points are.
The same maximum value occurs at $x=-1,1$, $y_{\max}=\frac{1}{e}$.

Q1biii $y_{\max}+d<0$, .: $d<-\frac{1}{e}$

Q1ci The tangent is a horizontal line: $y=\frac{1}{e}$

Q1cii Area $=\int_{-1}^{1}\left(\frac{1}{e}-x^2e^{-x^2}\right)dx\approx 0.3568$

Q1d $n=m^2e^{-m^2}$, distance $D=\sqrt{(m-0)^2+\left(m^2e^{-m^2}-e\right)^2}$
By CAS minimum distance ≈ 2.511, $m\approx 0.783$

Q2a By CAS $\frac{dy}{dx}=\frac{9(x-10)(x-30)}{2000}$

Q2b By reading the gradient graph $\frac{dy}{dx}$ is strictly decreasing for $x\in[0, 20]$.

Q2c $y=f(x)=\frac{3x(x-30)^2}{2000}+3$ for $x\in[a, 30]$

Q2d Average gradient of the hill $=\frac{0-\frac{3(10)(-20)^2}{2000}}{30-10}=-\frac{3}{10}$

Let $\frac{dy}{dx}=\frac{9(x-10)(x-30)}{2000}=-\frac{3}{10}$, $3x^2-120x+1100=0$

$x=\frac{120\pm\sqrt{14400-13200}}{6}=\frac{60\pm10\sqrt{3}}{3}\approx 14.23$ or 25.77

Q2ei At $A(a,b)$, $\frac{dy}{dx}=\frac{9(a-10)(a-30)}{2000}$

Q2eii $b=f(a)=\frac{3a(a-30)^2}{2000}+3$

Gradient of the straight section $=\frac{b-10}{a}=\frac{3a(a-30)^2-14000}{2000a}$

Let $\frac{3a(a-30)^2-14000}{2000a}=\frac{9(a-10)(a-30)}{2000}$

$\therefore$ $a\approx 11.12$, $b\approx 8.95$ $\therefore$ $A(11.12, 8.95)$ correct to 2 decimal places.

Q2eiii Gradient at $A=\frac{9(11.12-10)(11.12-30)}{2000}\approx -0.1$

Q3a 12

Q3b $t=0, 4, 6$

Q3c Maximum strength ≈ 1.76

Q3d Area $=\int_0^4\left(\sin\frac{\pi t}{3}+\sin\frac{\pi t}{6}\right)dt-\int_4^6\left(\sin\frac{\pi t}{3}+\sin\frac{\pi t}{6}\right)dt$

$=\left[-\frac{3}{\pi}\cos\frac{\pi t}{3}-\frac{6}{\pi}\cos\frac{\pi t}{6}\right]_0^4-\left[-\frac{3}{\pi}\cos\frac{\pi t}{3}-\frac{6}{\pi}\cos\frac{\pi t}{6}\right]_4^6$

$=\left(\frac{3}{2\pi}+\frac{6}{2\pi}+\frac{3}{\pi}+\frac{6}{\pi}\right)-\left(-\frac{3}{\pi}+\frac{6}{\pi}-\frac{3}{2\pi}-\frac{6}{2\pi}\right)=\frac{15}{\pi}$

Next 2 parts, *x* or *t* ?

Q3e $g(t)$ is the reflection of $f(t)$ in the t axis followed by a horizontal translation to the left by 6 units.

$\therefore$ $a=1$, $b=-1$, $c=-6$, $d=0$

Q3f $12k=2\times\frac{15}{\pi}$, $k=\frac{5}{2\pi}$

Q4a $\mu=\int_0^5\frac{4}{625}x(5x^3-x^4)dx=\frac{4}{625}\left[x^5-\frac{x^6}{6}\right]_0^5=\frac{10}{3}$

Q4b $\Pr(X>2)=\int_2^5\frac{4}{625}(5x^3-x^4)dx=\frac{4}{625}\left[\frac{5x^4}{4}-\frac{x^5}{5}\right]_2^5=0.91296$

Number of butterflies $=80\times0.91296\approx 73$

Q4c $\Pr(X\geq 4\mid X\geq 2)=\frac{\Pr(X\geq 4)}{\Pr(X\geq 2)}\approx\frac{0.26272}{0.91296}\approx 0.2878$

Q4d Let random variable L (cm) be the wingspan, $\mu=14.1$, $\sigma=2.1$

$\Pr(16<L<18)\approx 0.1512$

Q4e $\Pr(L<\ell)=0.05$, $\Pr\left(Z<\frac{\ell-14.1}{2.1}\right)=0.05$

$\frac{\ell-14.1}{2.1}\approx -1.6449$, $\ell=10.6$ cm is the greatest possible wingspan for a very small butterfly. OR invNorm$(5\%, 14.1, 2.1)$

Q4fi Binomial random variable X number of successes, $p=0.0527$, $n=36$ $\Pr(X\geq 3)\approx 0.2947$

Q4fii Define $v(n)=\text{binomCdf}(36, 0.0527, n, 36)$

$\Pr(X\geq 6)=0.01066>0.01$

$\Pr(X\geq 7)=0.00244<0.01$, $n=7$

Q4fiii $\text{E}(\hat{P})=p\approx 0.0527$, $\text{sd}(\hat{P})\approx\sqrt{\frac{0.0527(1-0.0527)}{36}}\approx 0.0372$

Q4fiv $\hat{P}$: $(0.0527-0.0372, 0.0527+0.0372)=(0.0155, 0.0899)$

X : $(36\times0.0155, 36\times0.0899)\approx(0.558, 3.2364)$, i.e. $(1, 3)$

$\Pr(1\leq X\leq 3)\approx 0.7380$

Q4g $0.0234=\hat{p}-1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, $0.0866=\hat{p}+1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Solve the simultaneous equations:

$\hat{p}\approx 0.055$, $0.0234=0.055-1.96\sqrt{\frac{0.055(1-0.055)}{n}}$, $n\approx 199.96$

The sample size was 200.

Q5a $f'(x) = -3x^2$, $f'(a) = -3a^2$, equation: $y = -3a^2x + 1 + 2a^3$ where $0 < a < 1$

Q5b At Q, $y = 0$, $x = \frac{1 + 2a^3}{3a^2}$

Q5c Let $-3a^2x + 1 + 2a^3 = 1 - x^3$, $x^3 - 3a^2x + 2a^3 = 0$, $x = -2a$

Q5d

$$A = \int_{-2a}^{1}\left(-3a^2x + 1 + 2a^3 - \left(1 - x^3\right)\right)dx + \int_{1}^{\frac{1+2a^3}{3a^2}}\left(-3a^2x + 1 + 2a^3\right)dx$$

$$= \left[\frac{-3a^2x^2}{2} + 2a^3x + \frac{x^4}{4}\right]_{-2a}^{1} + \left[\frac{-3a^2x^2}{2} + \left(1 + 2a^3\right)x\right]_{1}^{\frac{1+2a^3}{3a^2}}$$

$$= \left(\frac{-3a^2}{2} + 2a^3 + \frac{1}{4}\right) - \left(\frac{-12a^4}{2}\right)$$

$$+ \left(\frac{-3a^2}{2}\left(\frac{1+2a^3}{3a^2}\right)^2 + \left(1 + 2a^3\right)\left(\frac{1+2a^3}{3a^2}\right)\right) - \left(\frac{-3a^2}{2} + 1 + 2a^3\right)$$

$$= 6a^4 + \frac{\left(1 + 2a^3\right)^2}{6a^2} - \frac{3}{4} = \frac{80a^6 + 8a^3 - 9a^2 + 2}{12a^2}$$

or $\frac{20}{3}a^4 + \frac{2}{3}a - \frac{3}{4} + \frac{1}{6a^2}$

Q5e Let $\frac{dA}{da} = \frac{80}{3}a^3 + \frac{2}{3} - \frac{1}{3a^3} = 0$

$a^3 = \frac{1}{10}$, $a = \left(\frac{1}{10}\right)^{\frac{1}{3}}$ or approx. 0.4642

Q5f There is only one region bounded by the tangent and the graph of f^{-1}. The area of the region $\to 0$ as $b \to 1$.

There are two regions bounded by the tangent, the graph of f^{-1} and the **vertical** axis. In the latter case the two regions are the reflection of the two regions in **part e** in the line $y = x$. The image of $\left(a, 1 - a^3\right)$ after reflection is $\left(1 - a^3, a\right)$, $\therefore$ $b = 1 - a^3 = \frac{9}{10}$.

Q5g Tangent to f^{-1} at $x = 1$ is a vertical line.

Gradient of the tangent to f at $x = 1$ is -3, angle with horizontal axis $\theta = \tan^{-1}(-3)$.

Required acute angle $= \tan^{-1}(-3) + \frac{\pi}{2} = \tan^{-1}\left(\frac{1}{3}\right)$.

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