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2019 VCAA Math Methods Exam 2 (NHT) Solutions

Use CAS to save time

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	B	D	A	A	D	D	D	C	C

11	12	13	14	15	16	17	18	19	20
E	D	E	A	B	E	B	B	C	C

Q1 $\log_e x$ is defined for $x>0$ B

Q2 B

Q3 $f(-a)=8(-a)^3-14(-a)^2-a^2(-a)=0$, $-7a^3-14a^2=0$
$a=-2$ D

Q4 $y=2-\dfrac{11}{4+x}$ A

Q5 $b=\dfrac{2}{13}$, $\mathrm{E}(X)=2\left(\dfrac{3}{5}-\dfrac{2}{13}\right)+3\left(\dfrac{3}{5}\times\dfrac{2}{13}\right)=\dfrac{76}{65}$ A

Q6 $f(f(x))=(x^2+1)^2+1=\dfrac{185}{16}$, $(x^2+1)^2=\dfrac{169}{16}$, $x^2+1=\dfrac{13}{4}$,
$x^2=\dfrac{9}{4}$, $x=\dfrac{3}{2}$ D

Q7 $m=\int_1^3 \dfrac{2}{x}\,dx=2\log_e 3=\log_e 9$, $\therefore\ e^m=9$ D

Q8 $y=-\dfrac{m-1}{2}+1$ and $y=-\dfrac{3}{m}x+\dfrac{k}{m}$
Infinitely many solutions when $\dfrac{k}{m}=1$ and $\dfrac{3}{m}=\dfrac{m-1}{2}$
$m=-2$ or 3 and $k=-2$ or 3 respectively. D

Q9 $\Pr(RGR)+\Pr(GRG)=\dfrac{3}{5}\times\dfrac{2}{4}\times\dfrac{2}{3}+\dfrac{2}{5}\times\dfrac{3}{4}\times\dfrac{1}{3}=\dfrac{3}{10}$ C

Q10 $\int_0^{\frac{2}{3}} f(x)\,dx=1$, $\int_0^{\frac{2}{3}} k\left(18x^4+9x^3-8x^2-4x\right)dx=1$,
$k=-\dfrac{405}{308}$ C

Q11 E

Q12 $-(y+3)=\sqrt{2x+1}\rightarrow y'=\sqrt{x'}$
$\therefore\ x'=2x+1$, $y'=-(y+3)=-y-3$,
$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}2x+1\\-y-3\end{bmatrix}=\begin{bmatrix}2&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\-3\end{bmatrix}$ D

Q13 $f'(x)=3x^2-12(b-2)x+18=0$, two solutions when $\Delta>0$,
$(-12(b-2))^2-4(3)(18)>0$, $(b-2)^2>\dfrac{\sqrt{6}}{2}$ E

Q14 $x>0$, $y>0$, $\dfrac{x^2}{x+2}=y$, $x^2-yx-2y=0$, $x=\dfrac{y+\sqrt{y^2+8y}}{2}$ A

Q15 $\int_4^{10} 3f(x-2)\,dx=3\int_4^{10} f(x-2)\,dx=3\int_2^8 f(x)\,dx=9\log_3 13$ B

Q16 $g(f(x))\approx-\dfrac{1}{f(x)}\approx-\dfrac{1}{(x-1)^2+1}$, $g(f(0))\approx-\dfrac{1}{2}$ E

Q17
$\cos x-\dfrac{3}{8}\rightarrow\cos\dfrac{\pi x}{4}-\dfrac{3}{8}\rightarrow\dfrac{4}{3}\left(\cos\dfrac{\pi x}{4}-\dfrac{3}{8}\right)\rightarrow\dfrac{4}{3}\left(\cos\dfrac{-\pi x}{4}-\dfrac{3}{8}\right)$
$\rightarrow\dfrac{4}{3}\left(\cos\dfrac{-\pi x}{4}-\dfrac{3}{8}\right)+\dfrac{3}{2}=\dfrac{4}{3}\cos\dfrac{\pi x}{4}+1$ B

Q18 x-intercept: $2^{x-1}=8$, $x=4$; y-intercept: $y=\dfrac{15}{2}$
Shade region area $=\int_0^4\left(8-2^{x-1}\right)dx-\dfrac{1}{2}(4)\left(\dfrac{15}{2}\right)=17-\dfrac{15}{2\log_e 2}$ B

Q19 $\hat{p}=\dfrac{0.6668+0.8147}{2}=0.74075$
$0.74075+1.96\sqrt{\dfrac{0.74075(1-0.74075)}{n}}=0.8147$, $n\approx 135$ C

Q20 $f(0)=b^5=1$, $\therefore\ b=1$, $f(x)=(ax+1)^5$, $f'(x)=5a(ax+1)^4$
$\therefore\ f'(0)=5a$, $g'(1)=\dfrac{1}{f'(0)}=\dfrac{1}{5a}$ C

SECTION B

Q1a $(x-1)^2(x+2)^3-(x-1)^3(x+2)^3=0$,
$(x-1)^2(x+2)^3(1-(x-1))=0$, $x=2$, $y=(2-1)^2(2+2)^3=64$
$\therefore\ c=2$ and $d=64$

Q1b $f(x)>g(x)$ for $x\in(-\infty,-2)$

Q1bii Let $2xe^{-x^2}\left(1-x^2\right)=0$, $x=-1,0,1$ where stationary points are.
The same maximum value occurs at $x=-1,1$, $y_{\max}=\dfrac{1}{e}$.

Q1ci $f'(x)>0$ for $x\in\left(-\dfrac{1}{2},1\right)\cup(1,\infty)$

Q1cii $g'(x)>0$ for $x\in(-\infty,-2)\cup\left(-2,-\dfrac{1}{5}\right)\cup(1,\infty)$

Q1d $f(-2-m)=(-3-m)^3(-m)^3=m^3(3+m)^3$
$f(1+m)=m^3(3+m)^3$, $\therefore\ f(1+m)=f(-2-m)$ for all m

Q1e $g(x+h)=(x+h-1)^2(x+h+2)^3$, $h-1\le 0$ and $h+2>0$
$\therefore\ h\le 1$ and $h>-2$, i.e. $-2<h\le 1$

Q1f $f\left(-\dfrac{1}{2}\right)=-\left(\dfrac{3}{2}\right)^6$, $k>\left(\dfrac{3}{2}\right)^6$, i.e. $k>\dfrac{729}{64}$

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Q2a Amplitude $=16$ km/h, period $=\dfrac{2\pi}{\frac{\pi}{14}}=28$ min

Q2b Max $=20+16=36$ km/h, min $20-16=4$ km/h

Q2c $v(60)=20+16\sin\dfrac{60\pi}{14}\approx 32.5093$ km/h

Q2d Average value $=\dfrac{\int_0^{60}\left(20+16\sin\dfrac{\pi t}{14}\right)dt}{60}\approx 20.45$ km/h

Q2e $v_1(60)=v(60)\approx 32.5093$

Solve $28+18\sin\left(\dfrac{\pi(60-k)}{7}\right)\approx 32.5093$ for smallest $k\in R^+$,

$k\approx 3.4358$

Q2fi $28+18\sin\left(\dfrac{\pi(t-31.4358)}{7}\right)>38$, $t>60.748$, $t\approx 60.75$

Q2fii $v_1>38$ for $t\in(60.748, 65.123)$, period $=\dfrac{2\pi}{\frac{\pi}{7}}=14$,

proportion $=\dfrac{65.123-60.748}{14}=0.3125$, percentage $\approx 31\%$

Q2g $\dfrac{y-20}{16}=\sin\dfrac{\pi x}{14}$, $\dfrac{y'-28}{18}=\sin\dfrac{\pi(x'-k)}{7}$

$\therefore$ $y'=\dfrac{9}{8}y+\dfrac{11}{2}$, $x'=\dfrac{1}{2}x+k$

$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}$, $\therefore$ $a=\dfrac{1}{2}$, $c=k$, $b=\dfrac{9}{8}$, $d=\dfrac{11}{2}$

Q3a $\Pr(L<0)\approx 0.0062$

Q3b $\Pr(L>15)\approx 0.1056$

Q3ci $\Pr(C=0)\approx 0.006$, $\Pr(C=200)\approx 0.106$

$\therefore$ $\Pr(C=100)\approx 1-0.006-0.106=0.888$

Q3cii $\mathrm{E}(C)\approx 0\times 0.006+100\times 0.888+200\times 0.106=110$

Q3ciii $\mathrm{Var}(C)\approx 100^2\times 0.888+200^2\times 0.106-110^2=1020$

$\mathrm{sd}(C)=\sqrt{1020}\approx 32$

Q3di $\hat{p}=\dfrac{43}{1000}=0.043$, $\mathrm{sd}(\hat{P})\approx\sqrt{\dfrac{0.043(1-0.043)}{1000}}\approx 0.0064149$

$0.043\mp 1.96\times 0.0064149\approx 0.030, 0.056$

95% confidence interval $(0.030, 0.056)$

Q3dii The interval $(0.030, 0.056)$ suggests that p (at Mathsland Concert Hall) $\neq\hat{p}=0.043$ because p (at the Mathsland Concert Hall) is outside the interval.

Q3e $\mathrm{E}(M)=\int_0^\infty\dfrac{8x}{(x+2)^3}dx=8\int_0^\infty\dfrac{x+2-2}{(x+2)^3}dx$

$=8\int_0^\infty\left(\dfrac{x+2}{(x+2)^3}-\dfrac{2}{(x+3)^3}\right)dx=8\int_0^\infty\left(\dfrac{1}{(x+2)^2}-\dfrac{2}{(x+3)^3}\right)dx$

$=8\left[-\dfrac{1}{x+2}+\dfrac{1}{(x+2)^2}\right]_0^\infty=8\left(\dfrac{1}{2}-\dfrac{1}{4}\right)=2$

Q3fi $\Pr(M>15)=\int_{15}^\infty\dfrac{8}{(x+2)^3}dx=8\left[-\dfrac{1}{2(x+2)^2}\right]_{15}^\infty=\dfrac{4}{289}$

Q3fii Required probability $=\left(1-\dfrac{4}{289}\right)^9\left(\dfrac{4}{289}\right)\approx 0.0122$

Q3fiii $\Pr(M<20\mid M>15)=\dfrac{\Pr(15<M<20)}{\Pr(M>15)}$

$\Pr(15<M<20)=\int_{15}^{20}\dfrac{8}{(x+2)^3}dx=8\left[-\dfrac{1}{2(x+2)^2}\right]_{15}^{20}\approx 0.005576$

$\therefore$ $\Pr(M<20\mid M>15)\approx\dfrac{0.005576}{\Pr(M>15)}\approx 0.403$

Q4a Area of shaded region $=2\int_0^2 x(x-2)(x-4)dx=8$

Q4b Amount $=4\times 120000+4\times 100000=880000$ dollars

Q4c $x-4+\dfrac{4}{1+a}=x-2$, $\dfrac{4}{1+a}=2$, $a=1$

Q4d $p(x)=x\left(x-4+\dfrac{4}{1+a}\right)(x-4)$

By CAS $p'(x)=3x^2+\left(-16+\dfrac{8}{1+a}\right)x+16-\dfrac{16}{1+a}$

Let $p'(x)=0$, solve by CAS $x=\dfrac{4(2a+1)\pm 4\sqrt{a^2+a+1}}{3(a+1)}$

Q4e $p(x)=x\left(x-4+\dfrac{4}{1+a}\right)(x-4)\geq -4$ at the second turning point

where $p'(x)=0$, i.e. at $x=\dfrac{4(2a+1)+4\sqrt{a^2+a+1}}{3(a+1)}$

$p\left(\dfrac{4(2a+1)+4\sqrt{a^2+a+1}}{3(a+1)}\right)\geq -4$, $\therefore$ smallest $a\approx 0.716$ by CAS

Q4f $p(x)=x\left(x-4+\dfrac{4}{1+a}\right)(x-4)$

x-intercepts: $x=0, 4$ and $x-4+\dfrac{4}{1+a}=0$, i.e. $x=4-\dfrac{4}{1+a}$

By CAS $A_{total}(a)=\int_0^{4-\frac{4}{1+a}}p(x)dx-\int_{4-\frac{4}{1+a}}^{4}p(x)dx=\dfrac{64(1+2a+2a^3+a^4)}{3(1+a)^4}$

and let $A'_{total}(a)=0$, $a=1$

Q4g By CAS

$C_{total}(a) = 120000\int_0^{4-\frac{4}{1+a}} p(x)dx - 100000\int_{4-\frac{4}{1+a}}^{4} p(x)dx$

$= \dfrac{128000(5+10a+12a^3+6a^4)}{3(1+a)^4}$

and let $C'_{total}(a) = 0$, $a \approx 0.886$

Q5a $y = g(x) = 2\log_e x$, inverse: $x = 2\log_e y$, $y = g^{-1}(x) = e^{\frac{x}{2}}$

Q5b $y = f(x) = e^{\frac{x}{2}}$, $f'(x) = \dfrac{1}{2}e^{\frac{x}{2}} = 1$, $x = 2\log_e 2$, $y = 2$,

$A(2\log_e 2, 2)$

Q5c $\dfrac{y-2}{x-2\log_e 2} = -1$, $y = -x + 2\log_e 2 + 2$

Q5d B is the inverse of A, $B(2, 2\log_e 2)$.

Q5e $y = -x + 2\log_e 2 + 2$

Intercepts: $(0, 2\log_e 2 + 2)$ and $(2\log_e 2 + 2, 0)$

Area of shaded region

$= \dfrac{1}{2}(2\log_e 2 + 2)^2 - 2\int_0^{2\log_e 2}\left(-x + 2\log_e 2 + 2 - e^{\frac{x}{2}}\right)dx$

$= -2(\log_e 2)^2 - 4\log_e 2 + 6$

Q5f $p(x) = e^{kx}$, let $p'(x) = ke^{kx} = 1$, $x = \dfrac{1}{k}\log_e \dfrac{1}{k}$

$q(x) = \dfrac{1}{k}\log_e x$, let $q'(x) = \dfrac{1}{kx} = 1$, $x = \dfrac{1}{k}$

Since $\dfrac{1}{k}\log_e \dfrac{1}{k} = \dfrac{1}{k}$, .: $k = \dfrac{1}{e}$

Q5g $p'(x) = ke^{kx}$, $p'(0) = k$

$q(x) = \dfrac{1}{k}\log_e x = 0$, $x = 1$, $q'(x) = \dfrac{1}{kx}$, $q'(1) = \dfrac{1}{k}$

Parallel: $k = \dfrac{1}{k}$, $k = 1$

Please inform admin@itute.com re conceptual and/or mathematical errors

