2019 VCAA Specialist Mathematics Exam 2 Solutions

Use CAS to save time

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	C	D	C	D	A	B	E	B	C

11	12	13	14	15	16	17	18	19	20
E	A	D	A	E	A	E	D	C	B

Q1 $f'' \neq 0$ E

Q2 $f(x)=\frac{1}{2}x+2+\frac{17}{2x-8}$ C

Q3 $\ldots, \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$ D

Q4 $i^{1!}=i$, $i^{2!}=-1$, $i^{3!}=-1$, $i^{4!}$, ... , $i^{100!}=1$

Sum $=i-1-1+97=95+i$ C

Q5 Solve simultaneous equations $\frac{y}{x-2}=\tan\frac{\pi}{4}=1$ and

$\frac{y-1}{x-5}=\tan\frac{5\pi}{6}=-\frac{1}{\sqrt{3}}$, $y=\sqrt{3}$ D

Q6 $\arg\left(\frac{z^5}{w^4}\right)=5\text{Arg}(z)-4\text{Arg}(w)=\frac{5\pi}{2}-\pi=\frac{3\pi}{2}$

$\therefore$ $\text{Arg}\left(\frac{z^5}{w^4}\right)=-\frac{\pi}{2}$ A

Q7 $\frac{dx}{dt}=3\cos t$, $\frac{dy}{dt}=-4\sin t$

Length $=\int_0^\pi \sqrt{(3\cos t)^2+(-4\sin t)^2}\,dt=\int_0^\pi \sqrt{9+7\sin^2 t}\,dt$ B

Q8 Let $u=2x+1$, $\int_1^5 (2x-1)\sqrt{2x+1}\,dx=\frac{1}{2}\int_3^{11}(u-2)u^{\frac{1}{2}}\,du$ E

Q9 Gradient ≈ 1 at $(0,0)$ and gradient ≈ 0 at $\left(\frac{\pi}{2},\pi\right)$ B

Q10 Radius $h\tan 60°=\sqrt{3}h$, $V=\frac{1}{3}\pi\left(\sqrt{3}\,h\right)^2 h=\pi h^3$

$\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dh}}=\frac{1.5}{3\pi(0.5)^2}\approx 0.64$ C

Q11 $\left(\frac{a-3}{2},\frac{1+b}{2},\frac{-2-1}{2}\right)=\left(-5,\frac{3}{2},c\right)$

$\therefore$ $a=-7$, $b=2$, $c=-\frac{3}{2}$ E

Q12 Let $\tilde{a}=\tilde{i}+\tilde{j}-\tilde{k}$ and $\tilde{b}=m\tilde{i}+n\tilde{j}+p\tilde{k}$.

$\hat{b}=\frac{1}{\sqrt{m^2+n^2+p^2}}\left(m\tilde{i}+n\tilde{j}+p\tilde{k}\right)$, $\tilde{a}.\hat{b}=\frac{m+n-p}{\sqrt{m^2+n^2+p^2}}$

Vector resolute

$=\left(\tilde{a}.\hat{b}\right)\hat{b}=\frac{m+n-p}{m^2+n^2+p^2}\left(m\tilde{i}+n\tilde{j}+p\tilde{k}\right)=2\tilde{i}-3\tilde{j}+\tilde{k}$ A

Q13 $\tilde{F}_1=2\tilde{j}$, $2\tilde{j}+\tilde{F}_2=3\left(\sqrt{3}\tilde{i}+\tilde{j}\right)$, $\tilde{F}_2=3\sqrt{3}\tilde{i}+\tilde{j}$

$\tilde{F}_1.\tilde{F}_2=2\times 2\sqrt{7}\cos\theta=2$, $\theta=\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)$ which is acute D

Q14 Total mass $=7$, net force $=3g$, $a=\frac{3g}{7}$

$g-T=\frac{3g}{7}$, $T=\frac{4g}{7}$ A

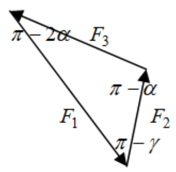
Q15 Projectile motion under a constant force (not in the direction of motion) E

Q16 $a=v\frac{dv}{dx}=e^x\sin x\left(e^x\cos x+e^x\sin x\right)=e^{2x}\left(\frac{1}{2}\sin 2x+\sin^2 x\right)$ A

Q17 E

$\frac{F_1}{\sin(\pi-\alpha)}=\frac{F_2}{\sin(\pi-2\alpha)}$,

$\frac{F_1}{F_2}=\frac{\sin(\pi-\alpha)}{\sin(\pi-2\alpha)}=\frac{\sin\alpha}{2\sin\alpha\cos\alpha}=\frac{1}{2}\sec\alpha$



Q18 $\Pr(Z<z)=0.99$, $z\approx 2.32635$

$\left(65-2.32635\times\frac{4}{\sqrt{36}}, 65+2.32635\times\frac{4}{\sqrt{36}}\right)\approx(63.4, 66.6)$ D

Q19 $\text{E}(aX+bY)=a\text{E}(X)+b\text{E}(Y)$, $4a+4b=8$

$\text{var}(aX+bY)=a^2\,\text{var}(X)+b^2\,\text{var}(Y)$, $9a^2+9b^2=90$

$\therefore$ $a=3$ and $b=-1$ OR $a=-1$ and $b=3$ C

Q20 $\text{sd}\left(\overline{X}\right)=\frac{0.2887}{\sqrt{100}}=0.02887$

$p=\Pr\left(\overline{X}\leq 0.47251\,|\,\mu=0.5\right)\approx 0.1704$ B

SECTION B

Q1a $t \in \left[0, \frac{\pi}{2}\right)$, .: $y \geq 0$, $\sec t = \sqrt{1+\tan^2 t} = \sqrt{1+y^2} = x - 1$

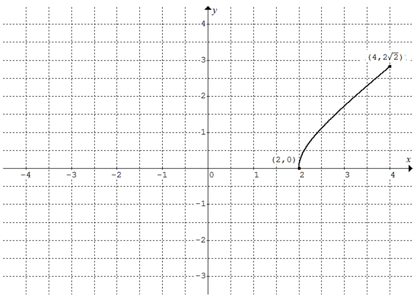
$1 + y^2 = (x-1)^2$, $y = \sqrt{x^2 - 2x}$

Q1b The domain is $[2, \infty)$ and the range is $[0, \infty)$

Q1ci $\frac{dx}{dt} = \sin t \sec^2 t$, $\frac{dy}{dt} = \sec^2 t$, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{\sin t}$

Q1cii $t \in \left[0, \frac{\pi}{2}\right)$, $t \to \frac{\pi}{2}$, $\frac{dy}{dx} \to 1$

Q1d

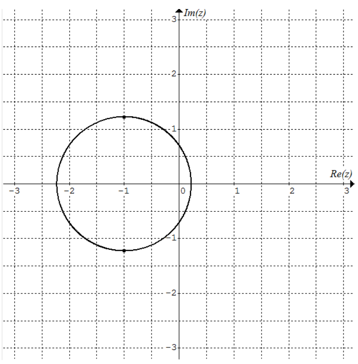


Q1e $\int_0^{2\sqrt{2}} \pi x^2 dy = \int_0^{2\sqrt{2}} \pi (\sec t + 1)^2 dy = \int_0^{\tan^{-1} 2\sqrt{2}} \pi (\sec t + 1)^2 \frac{dy}{dt} dt$

$= \int_0^{\tan^{-1} 2\sqrt{2}} \pi (\sec t + 1)^2 \sec^2 t \, dt$

Q2ai $z = \frac{-4 \pm \sqrt{16-40}}{4} = \frac{-4 \pm 2\sqrt{6}i}{4} = -1 \pm \frac{\sqrt{6}}{2} i$

Q2aii



Q2bi $m = 1$, $n = \frac{\sqrt{6}}{2}$.

Q2bii $(x+1)^2 + y^2 = \frac{3}{2}$

Q2biii The circle $|z+1| = \frac{\sqrt{6}}{2}$ is shown above.

Q2c $z = -1 \pm \frac{1}{2}\sqrt{4-2d}$, .: $z + 1 = \pm\sqrt{\frac{2-d}{2}}$

$|z+1| \leq \frac{\sqrt{6}}{2}$, $\left|\sqrt{\frac{2-d}{2}}\right| \leq \sqrt{\frac{3}{2}}$, $|2-d| \leq 3$, .: $-1 \leq d \leq 5$ or $d \in [-1, 5]$

Q2d $z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

$z + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$, $\left|z + \frac{b}{2a}\right| = \left|\frac{\sqrt{b^2-4ac}}{2a}\right|$

$p = \frac{b}{2a}$ and $q = \left|\frac{\sqrt{b^2-4ac}}{2a}\right|$

Q3ai $\frac{dt}{dP} = \frac{1}{k}.\frac{1}{P}$, $t = \frac{1}{k}\int \frac{1}{P} dP$, $kt = \log_e P + c$ where $P > 0$, $t \geq 0$

.: $ka = \log_e r + c$, $kb = \log_e s + c$, $ka - kb = \log_e r - \log_e r = \log_e\left(\frac{r}{s}\right)$

.: $k = \frac{1}{a-b}\log_e\left(\frac{r}{s}\right)$ where a, b, r and $s > 0$

Q3aii When $a - b > 0$ and $\frac{r}{s} > 1$, i.e. $a > b$ and $r > s$, then $k > 0$

Q3bi $\frac{dQ}{dt} = \frac{e^t}{e^Q}$, $\int e^Q dQ = \int e^t dt$

Q3bii $e^Q = e^t + c$, $Q = 1$ when $t = 0$, .: $c = e - 1$ and $Q = \log_e(e^t + e - 1)$

Q3biii $\frac{dQ}{dt} = \frac{e^t}{e^Q} = \frac{e^t}{e^t + e - 1}$

$\frac{d^2Q}{dt^2} = \frac{e^t(e^t + e - 1) - e^t.e^t}{(e^t + e - 1)^2} = \frac{e^t(e-1)}{(e^t + e - 1)^2} \neq 0$ for $t \geq 0$

.: the graph of Q does not have a point of inflection.

Q4a $\overrightarrow{AB} = (4-2)\tilde{i} + (-2+1)\tilde{j} + (1-3)\tilde{k} = 2\tilde{i} - \tilde{j} - 2\tilde{k}$

$\overrightarrow{DC} = (a-4)\tilde{i} + (b-3)\tilde{j} + (c+1)\tilde{k}$

$ABCD$ is a parallelogram, .: $\overrightarrow{AB} = \overrightarrow{DC}$, .: $a = 6$, $b = 2$ and $c = -3$.

Q4b $\overrightarrow{AD} = 2\tilde{i} + 4\tilde{j} - 4\tilde{k}$, $\cos \angle BAD = \frac{AB.AD}{|AB|.|AD|} = \frac{4}{9}$

Q4c Height of parallelogram $= |AB| \sin\left(\cos^{-1}\left(\frac{4}{9}\right)\right) = \frac{\sqrt{65}}{3}$

Area of parallelogram $=$ height $\times |AD| = \frac{\sqrt{65}}{3} \times 6 = 2\sqrt{65}$

Q4d $\left(6\tilde{i}+2\tilde{j}+5\tilde{k}\right).\overrightarrow{AB}=0$, $\left(6\tilde{i}+2\tilde{j}+5\tilde{k}\right).\overrightarrow{AD}=0$

.: $6\tilde{i}+2\tilde{j}+5\tilde{k}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

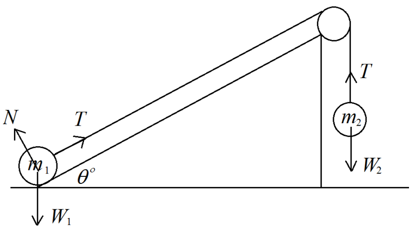
$\overrightarrow{AB}$ and $\overrightarrow{AD}$ define the base parallelogram

.: unit vector $\hat{u}=\frac{1}{\sqrt{65}}\left(6\tilde{i}+2\tilde{j}+5\tilde{k}\right)$ is perpendicular to the base of the pyramid.

Q4e $\overrightarrow{BP}=-2\tilde{j}+8\tilde{k}$, height of the pyramid $=\overrightarrow{BP}.\hat{u}=\frac{36}{\sqrt{65}}$

Volume of the pyramid $=\frac{1}{3}\times\frac{36}{\sqrt{65}}\times 2\sqrt{65}=24$

Q5a



Q5b $m_1 g\sin\theta=m_2 g$, .: $\sin\theta=\frac{m_2}{m_1}$

Q5ci $0<\theta<\frac{\pi}{2}$ and $m_1 g\sin\theta<m_2 g$, $0<\theta<\sin^{-1}\left(\frac{m_2}{m_1}\right)$

Q5cii Assuming $m_1 g\sin\theta<m_2 g$, $a=\frac{(m_2-m_1\sin\theta)g}{m_1+m_2}$

Q5d Just after travelling 2 m up along the plane:

$u=0$, $a=\frac{(m_2-m_1\sin\theta)g}{m_1+m_2}=\frac{\left(\frac{1}{2}m_1-\frac{1}{4}m_1\right)g}{m_1+\frac{1}{2}m_1}=\frac{g}{6}$, $s=2$

$v^2=2\times\frac{g}{6}\times 2=\frac{2g}{3}$

Now m_1 starts to slow down, $a=-g\sin\theta=-\frac{g}{4}$, $u^2=\frac{2g}{3}$,

$v=0$, $0=u^2+2as=\frac{2g}{3}-2\times\frac{g}{4}s$, .: $s=\frac{4}{3}$

Maximum distance $=2+\frac{4}{3}=\frac{10}{3}$ m

Q6a $\text{sd}\left(\overline{X}\right)=\frac{15}{\sqrt{50}}=\frac{3}{\sqrt{2}}$, $\Pr\left(370<\overline{X}<375\right)\approx 0.490789$

$\Pr(\text{at least one})=1-\Pr(\text{none})\approx 1-(1-0.490789)^2\approx 0.741$

Q6b Let $D=\overline{X}_1-\overline{X}_2=2$, $\text{E}(D)=0$,

$\text{sd}(D)=\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2+\left(\frac{3}{\sqrt{2}}\right)^2}=3$

$\Pr(|D|<2)=\Pr(-2<D<2)\approx 0.495$

Q6c $H_0:\ \mu=375$, $H_1:\ \mu\neq 375$

Q6d $\text{sd}\left(\overline{X}\right)=\frac{15}{\sqrt{100}}=1.5$

For two tailed test, $p=2\times\Pr\left(\overline{X}<372\mid\mu=375\right)\approx 0.046$

Q6e Since $p<0.05$, the machine is not working properly at the 5% level of significance for a two-tailed test.

Q6f $2\times\Pr\left(\overline{X}<\min x\mid\mu=375\right)<0.05$, $\min x\approx 372.1$

Please inform mathline@itute.com re conceptual and/or mathematical errors