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2020 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

Modelling Task

Time allowed: 2.5 hours plus

Application Task: Parts I, II and III
Modelling Task: Parts I and II only OR Parts I and III only

Theme: Slides

Slides can be found in family backyards, neighbourhood playgrounds, swimming pools and fun parks. The general features of most slides are a steep ladder for climbing up to a platform, and a sloping plane for sliding down from the platform to a level just above the ground or water. The sliding plane can be straight or curved, and the curved plane can have dips and humps added depending on the level of excitement required.

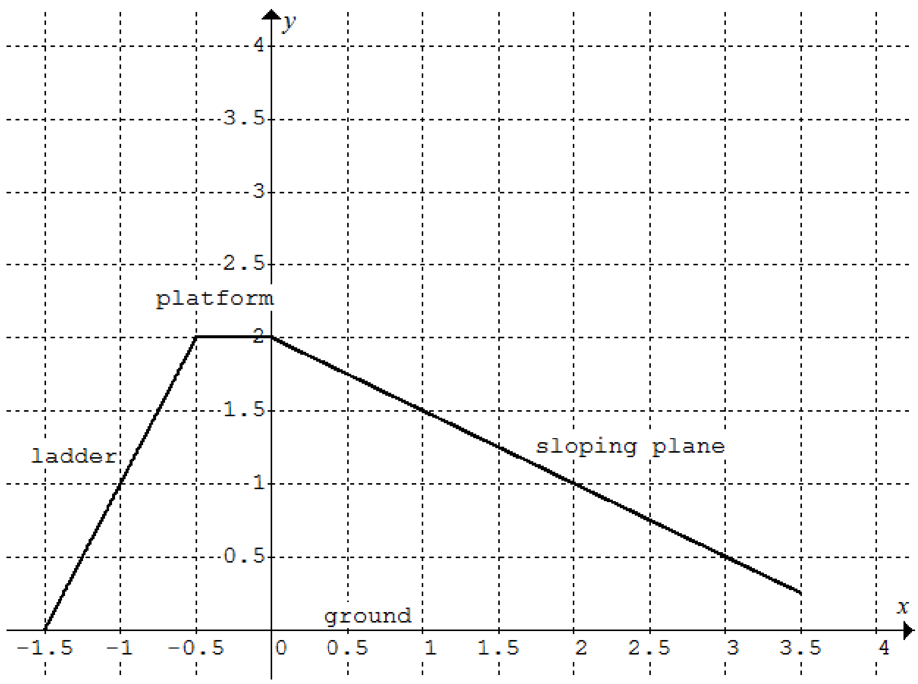
Assumed knowledge: Functions, transformations, algebra, differentiation and use of CAS

Specifications:
For this application task a slide is considered **safe** if the sloping plane does not exceed a gradient of 1 **in magnitude** at any point along its entire length.
All length measurements are in metres.

Answer this question. In this task the same scales for both axes in graph sketching are used. Why?

Part I (80 – 90 minutes)
Using a single function to model the sloping plane

The following graph shows the profile of a simple slide.



a. Write down the equation of each section (ladder, platform, sloping plane) of the simple slide.

b. Write a piecewise function $f(x)$ to model the profile of the slide (all three sections).

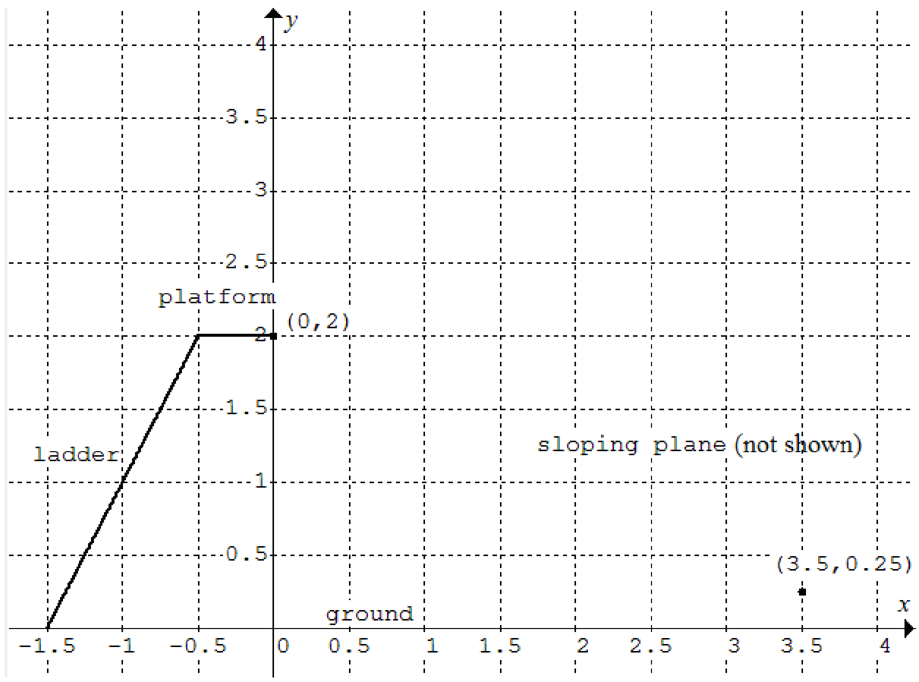
The coordinates of the highest point and the lowest point of the sloping plane are $(0, 2)$ and $(3.5, 0.25)$ respectively.

c. Instead of a straight sloping plane the designer uses a curved sloping plane joining the highest point to the lowest point.

The equation of the curve is $y = \dfrac{a}{x-b}$ where $a, b \in R$.

Show that $a = 1$ and $b = -0.5$.

d. Sketch accurately the graph of the curved sloping plane in the diagram below.



e. Determine the average gradient of the design in part c.

f. The design in part c is considered unsafe. Explain with calculations.
(Refer to the specifications at the beginning of the task.)

g. The designer attempts to make the design in part c safe by a horizontal dilation of factor 2, keeping the highest point and the lowest point of the sloping plane the same.

He replaces x with $\frac{x}{2}$ and writes the new equation as $y = \frac{c}{\frac{x}{2} - d}$.

Show that $c = 0.5$ and $d = -0.25$.

Discuss the effects of the designer's attempt on the sloping plane if any.

h. The designer attempts a second time to make the design in part c safe by a vertical dilation of factor $\frac{1}{2}$, keeping the highest point and the lowest point of the sloping plane the same.
He replaces y with $\frac{y}{\frac{1}{2}}$ $(=2y)$ and writes the new equation as $2y=\frac{e}{0.5x-f}$.
Show that $e=1$, $f=-0.25$.
Discuss the effects of the designer's second attempt on the sloping plane if any.

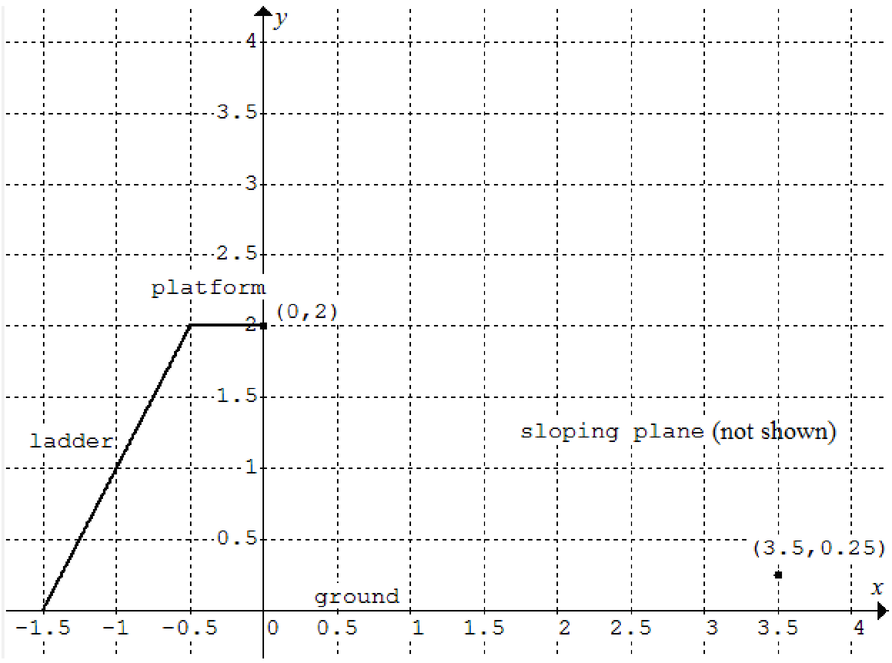
i. Explain your findings in part g and part h.

j. The designer attempts a third time to make the design in part c safe by choosing an exponential function, keeping the highest point and the lowest point of the sloping plane the same.

He uses the equation $y=Ae^{kx}$ where $A, k \in R$.
Show that $A=2$ and $k \approx -0.594126$.

k. The design in the third attempt is still unsafe but only slightly exceeding the safe limit for a small section, $0 < x < p$, of the sloping plane. Determine the value of p correct to 2 decimal places.

l. Sketch accurately the graph of the curved sloping plane of the third attempt (part j) in the diagram below.



m. Write a piecewise function to represent the profile of the slide (ladder, platform, sloping plane) in part l above.

n. Write another piecewise function to represent the profile of a slide (ladder, platform, sloping plane) which is half the size of the one in part l above for younger kids.
It is half in length in both *x*, *y* directions.

Part II (80 – 90 minutes)
Using two curves joined end to end to model the sloping plane

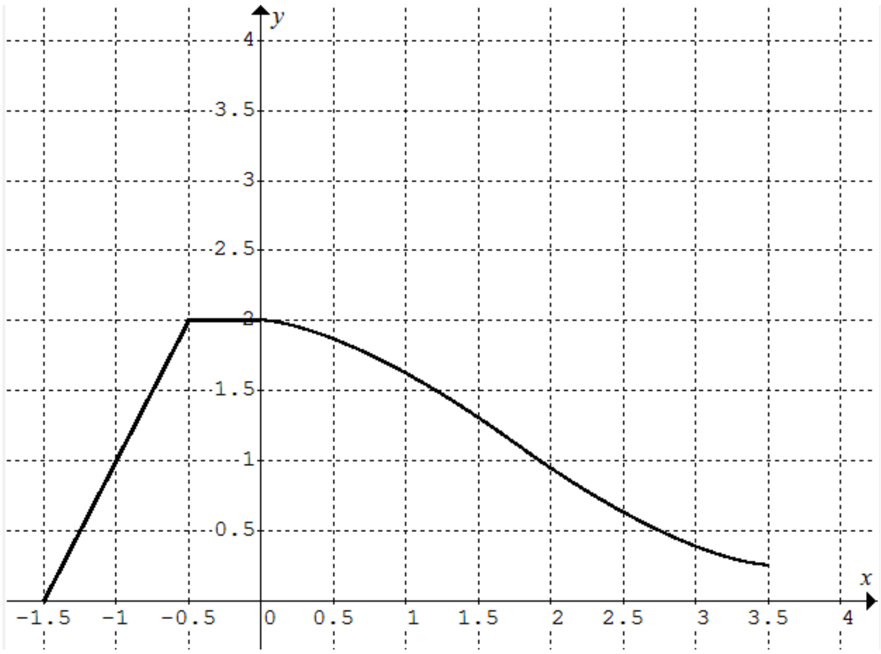
The designer wants to redesign the slide to meet the safety requirement (magnitude of gradient of the sloping plane does not exceed 1). The ladder and platform remain the same as in Part I.

The highest point and the lowest point are $(0, 2)$ and $(3.5, 0.25)$ respectively, same as in Part I.

He decides to use curves of the type $y = ax^n + b$ where $n \in Q$, and the maximum gradient is 0.75 in magnitude at any point along the sloping plane.

The sloping plane consists of two curves joined at the midpoint between $(0, 2)$ and $(3.5, 0.25)$.
The lower half is the transformation of the upper half of the sloping plane.

The graph below shows the profile of the sloping plane.



a. Show by calculations that the midpoint of the sloping plane is $(1.75, 1.125)$.

b. Show that the equation of the upper half of the sloping plane is $y = -0.378x^{1.500} + 2.000$, correct to 3 decimal places.

c. Show that the equation of the lower half of the sloping plane is $y = 0.378(3.500 - x)^{1.500} + 0.250$.

d. Show that the two halves are joined smoothly at the midpoint of the sloping plane.

e. Investigate and find the range of n values which gives a safe sloping plane similar in curvature to the one shown above.

f. Explain/discuss why cubic, quartic etc. power functions x^n in $y = ax^n + b$ are not suitable curves for the sloping plane.

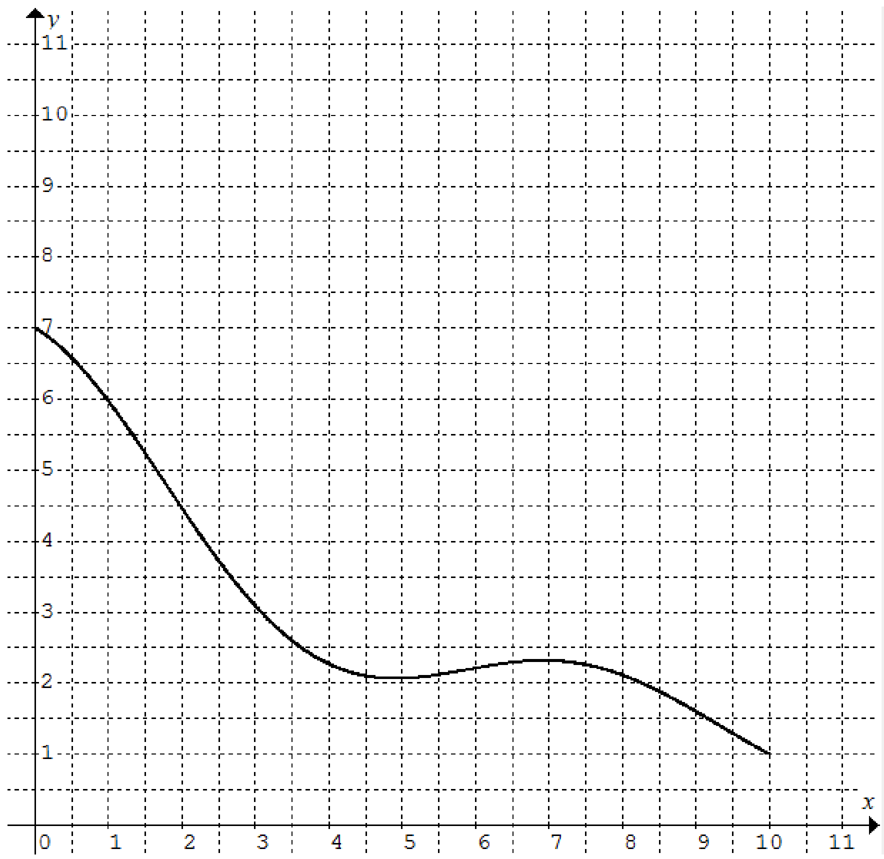
g. Write a piecewise function for the slide (ladder, platform, upper half, lower half).

Part III (80 – 90 minutes)
Using the product of two functions to model the sloping plane

The designer is contracted to design a large water slide next to a swimming pool.

The specifications written in the contract are:
** the sloping plane is fun and exciting
** the gradient of the sloping plane meets the safety standard
** the sloping plane spans (horizontally) 10.00 m
** the highest point (the start) of the sloping plane is 7.00 m above the water level
** the lowest point (the end) of the sloping plane is 1.00 m above the water level

He decides to model the sloping plane using a curve which is the product of a linear function and a circular function.
Initially the equation of the curve is $y = a(1 - bx)(2 + \cos(kx)) + c$ where $a = 2$, $b = 0.1$, $k = 0.7$ and $c = 1$.
The profile of the sloping plane (the curve) is shown in the graph below.
The water level is $y = 0$.



a. By studying the above graph (no calculations required) discuss whether the model meets the specifications.

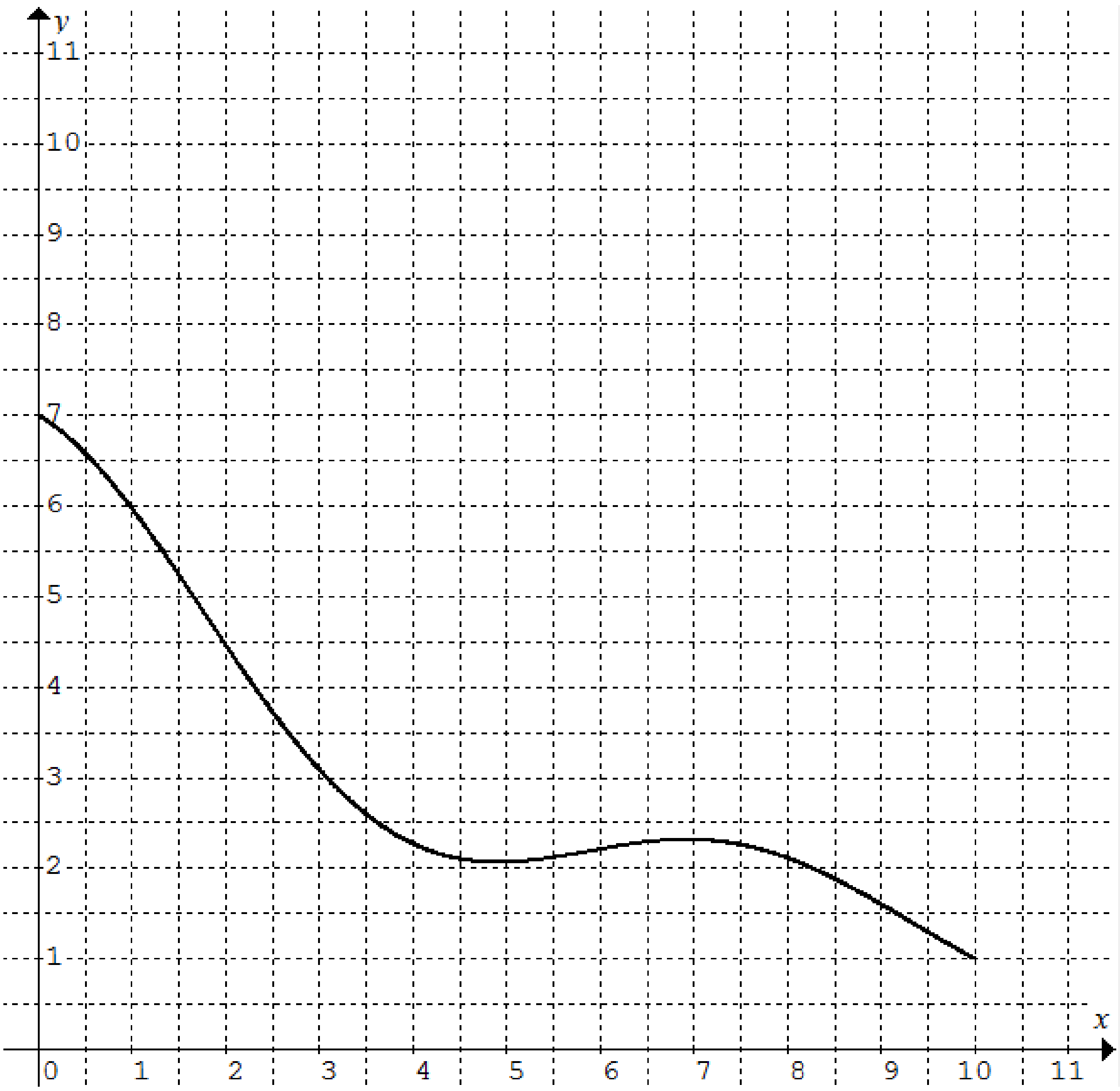
b. Keeping $b = 0.1$, $k = 0.7$ and $c = 1$ in the model, investigate the effects of varying the value of parameter a.
Suggestion: Graph the equations for 5 selected values of a where $1.6 \le a \le 2.4$.
Include $a = 2$ for comparison in your selection.
Comment on your findings. No graph sketching is required.

c. Keeping $a = 2$, $b = 0.1$ and $k = 0.7$ in the model, investigate the effects of varying the value of parameter c.
Suggestion: Graph the equations for 5 selected values of c where $0.6 \le c \le 1.4$.
Include $c = 1$ for comparison in your selection.
Comment on your findings. No graph sketching is required.

d. Keeping $a = 2$, $k = 0.7$ and $c = 1$ in the model, investigate the effects of varying the value of parameter b.
Suggestion: Graph the equations for 5 selected values of b where $0.06 \le b \le 0.14$.
Include $b = 0.1$ for comparison in your selection.
Comment on your findings. No graph sketching is required.

e. Keeping $a = 2$, $b = 0.1$ and $c = 1$ in the model, investigate the effects of varying the value of parameter k.
Suggestion: Graph the equations for 5 selected values of k where $0.4 \le k \le 0.8$.
Include $k = 0.7$ for comparison in your selection.
Comment on your findings.

The graph for $k = 0.7$ is shown in the axes below.
Sketch accurately the four other graphs on the same axes and label each one with the corresponding k values.



f. Select an appropriate value for each of the four parameters a, b, k and c such that the sloping plane (the curve) best meets the specifications in the contract.

g. Consider **your chosen** sloping plane in part f.
Find the x-coordinates of the points on the slope where the gradient has a magnitude of 1.
Hence state the interval(s) where the gradient of the slope fails to satisfy the safety requirement.

h. Discuss the relationship between safety and fun/excitement in the context of designing the sloping plane in Part III.

End of Application/Modelling Task