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2020 NSW ESA Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	C	D	B	C	D	A	C	B	A

Q1 $x^2-2x-3=(x-3)(x+1)\ge 0$, $x\le -1$, $x\ge 3$ **A**

Q2 $f(x)=1+\sqrt{x}$, dom $x\ge 0$, range $y\ge 1$

For $f^{-1}(x)$, dom $x\ge 1$, range $y\ge 0$ **C**

Q3 $\int \frac{1}{4x^2+1}dx=\frac{1}{4\times\frac{1}{2}}\int\frac{\frac{1}{2}}{x^2+\left(\frac{1}{2}\right)^2}dx=\frac{1}{2}\tan^{-1}(2x)+c$ **D**

Q4 $(2\tilde{i}+3\tilde{j})+(3\tilde{i}-2\tilde{j})+(4\tilde{i}-3\tilde{j})=9\tilde{i}-2\tilde{j}$, $\sqrt{9^2+(-2)^2}=\sqrt{85}$ **B**

Q5 $p(x)=(x-a)^2(x^2+x+1)$, x^2+x+1 cannot be factorised

.: only one x-intercept at $x=a$. Coefficient of x^4 is positive. **C**

Q6 $\tilde{a}=\tilde{v}+\tilde{b}$ **D**

Q7 At $x=0$, $\frac{dy}{dx}=0$; at $x=1$, $y=1$, $\frac{dy}{dx}=-\frac{1}{4}$; at $x=2$, $y=1$, $\frac{dy}{dx}=-\frac{1}{2}$ **A**

Q8 ${}^{10}C_6\times{}^6P_4=\frac{10.9.8.7.6.5}{6.5.4.3.2.1}\times 6.5.4.3=\frac{10!}{4!2!}$ **C**

Q9 Let $\tilde{a}=4\tilde{i}+8\tilde{j}$ be the projection of $\tilde{b}=6\tilde{i}+7\tilde{j}$ onto $y=2x$ and $\tilde{c}=\tilde{a}-\tilde{b}=-2\tilde{i}+\tilde{j}$.

The position vector of point A is $\tilde{a}+\tilde{c}=2\tilde{i}+9\tilde{j}=\begin{pmatrix}2\\9\end{pmatrix}$ **B**

Q10 $\frac{dP}{dt}=l^2k^2\in R^+$, $\frac{dQ}{dt}=\frac{1}{m^2}\times\frac{dP}{dt}=\frac{l^2k^2}{m^2}\in R^+$ **A**

Section II

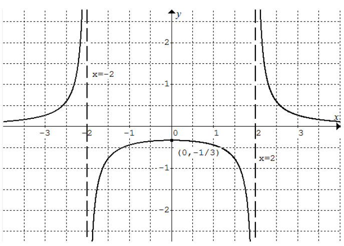
Q11ai $P(2)=2^3+3(2)^2-13(2)+6=0$

Q11aii $P(x)=(x-2)(x^2+5x-3)$ by long division or comparing coefficients

Q11b $\begin{pmatrix}a\\-1\end{pmatrix}\cdot\begin{pmatrix}2a-3\\2\end{pmatrix}=0$, $a(2a-3)+(-1)2=0$, $2a^2-3a-2=0$,

$(2a+1)(a-2)=0$.: $a=-\frac{1}{2}$ or 2

Q11c



Q11d $A\sin(x+\alpha)=A\sin x\cos\alpha+A\cos x\sin\alpha$

.: $A\cos\alpha=\sqrt{3}$ and $A\sin\alpha=3$.: $\tan\alpha=\sqrt{3}$, $\alpha=\frac{\pi}{3}$, $A=2\sqrt{3}$

.: $2\sqrt{3}\sin\left(x+\frac{\pi}{3}\right)=\sqrt{3}$, $x+\frac{2\pi}{6}=\frac{5\pi}{6},\frac{13\pi}{6}$

.: $x=\frac{\pi}{2},\frac{11\pi}{6}$

Q11e $\frac{dx}{dy}=e^{-2y}$, $x=\int e^{-2y}dy=\frac{e^{-2y}}{-2}+c=-\frac{1}{2e^{2y}}+c$

Q12a $n=1$, $1\times 2=1^2(1+1)$

Assume $1\times 2+2\times 5+3\times 8+\cdots+k(3k-1)=k^2(k+1)$ for $n=k$

For $n=k+1$:

$1\times 2+2\times 5+3\times 8+\cdots+k(3k-1)+(k+1)(3(k+1)-1)$

$=k^2(k+1)+(k+1)(3(k+1)-1)=(k+1)^2((k+1)+1)$

.: the statement is true. Hence it is true for all $n\ge 1$.

Q12bi $\mathrm{E}(X)=np=100\times\frac{3}{5}=60$

Q12bii $\mathrm{Var}(X)=np(1-p)=60\times\frac{2}{5}=24$, $\sigma_X=\sqrt{24}\approx 5$

Q12biii Normal approximation, $\mu=60$, $\sigma\approx 5$

$P(55<X<65)\approx P(-1<Z<1)\approx 0.68$

Q12c Select 3 topics from 8 in ${}^8C_3=56$ ways,

$400\div 56=7$ remainder 8.

.: at least $7+1=8$ students passed exactly the same three topics.

Q12d $\int_0^{\frac{\pi}{2}}\cos 5x\sin 3x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin 8x-\sin 2x)dx$

$=\frac{1}{2}\left[-\frac{\cos 8x}{8}+\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}=-\frac{1}{2}$

Q12e $\frac{dy}{dx}=-\frac{x}{y}$ and $(1,0)$, $\int y\,dy=-\int x\,dx$, $\frac{y^2}{2}=-\frac{x^2}{2}+c$

.: $x^2+y^2=1$, a circle centred at $(0,0)$ and has radius $r=1$.

Q13ai $\frac{d}{d\theta}(\sin^3\theta)=\frac{d(\sin\theta)^3}{d(\sin\theta)}\times\frac{d(\sin\theta)}{d\theta}=3\sin^2\theta\cos\theta$

Q13aii Let $x=\tan\theta$, $\frac{dx}{d\theta}=\sec^2\theta$

$\int_0^1\frac{x^2}{(1+x^2)^{\frac{5}{2}}}dx=\int_0^{\frac{\pi}{4}}\frac{\tan^2\theta}{(1+\tan^2\theta)^{\frac{5}{2}}}\sec^2\theta\,d\theta=\int_0^{\frac{\pi}{4}}\frac{\tan^2\theta}{(\sec^2\theta)^{\frac{5}{2}}}\sec^2\theta\,d\theta$

$=\int_0^{\frac{\pi}{4}}\frac{\tan^2\theta}{\sec^3\theta}d\theta=\int_0^{\frac{\pi}{4}}\sin^2\theta\cos\theta\,d\theta=\left[\frac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{4}}=\frac{1}{6\sqrt{2}}=\frac{\sqrt{2}}{12}$

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Q13b At the intersection, $0 < x < \frac{\pi}{2}$, $\sin x = \cos 2x$

.: $\sin x = 1 - 2\sin^2 x$, $2\sin^2 x + \sin x - 1 = 0$,

$(2\sin x - 1)(\sin x + 1) = 0$, .: $x = \frac{\pi}{6}$

$$V = \int_0^{\frac{\pi}{6}} \pi\left(\cos^2 2x - \sin^2 x\right)dx = \pi\int_0^{\frac{\pi}{6}}\left(\frac{1}{2}(1+\cos 4x) - \frac{1}{2}(1-\cos 2x)\right)$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{6}}(\cos 4x + \cos 2x)dx = \frac{\pi}{2}\left[\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{6}} = \frac{3\sqrt{3}\,\pi}{16}$$

Q13ci Let $\theta = \cos^{-1} x$, $f(x) = \tan\left(\cos^{-1} x\right)$

$$f'(x) = \frac{d\tan\theta}{d\theta}.\frac{d\theta}{dx} = \sec^2\left(\cos^{-1} x\right).\frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\cos^2\left(\cos^{-1} x\right)}.\frac{-1}{\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

$$g(x) = \frac{\sqrt{1-x^2}}{x},$$

$$g'(x) = \frac{x\frac{-x}{\sqrt{1-x^2}} - \sqrt{1-x^2}}{x^2} \times \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = \frac{-x^2 - \left(1-x^2\right)}{x^2\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

.: $f'(x) = g'(x)$

Q13cii $f'(x) = g'(x)$, $f(x) = \int g'(x)dx = g(x) + c$

Since $g(1) = \frac{0}{1} = 0$ and $f(1) = \tan\left(\cos^{-1} 1\right) = \tan 0 = 0$

.: $c = 0$ and $f(x) = g(x)$

Q14ai $(x+1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n-1}x + \binom{n}{n}$

$$(x+1)^{2n} = \binom{2n}{0}x^{2n} + \binom{2n}{1}x^{2n-1} + \cdots + \binom{2n}{n}x^{2n-n} + \cdots + \binom{2n}{2n-1}x + \binom{2n}{2n}$$

Since $(x+1)^{2n} = (x+1)^n(x+1)^n$.: same coefficients of x^n on both sides of the identity.

$$.:\ \binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{0}\binom{n}{n}$$

$$.:\ \binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 \qquad \text{Note: } \binom{n}{k} = \binom{n}{n-k}$$

Q14aii Chosen from a club of $2n$ members of n women and n men, there are $\binom{n}{0}^2$ ways for group size of 0; $\binom{n}{1}^2$ ways for group size of 2 etc.

.: total number of ways $= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$

Q14aiii Consider the size $2m$ group, $0 \le m \le n$.
There are m ways of selecting a female leader and m ways of selecting a male leader.
.: number of ways to choose the even number ($2m$) of people and then the leaders is $m^2\binom{n}{m}^2$.

.: total number of ways for $0 \le m \le n$ is

$$0^2\binom{n}{0}^2 + 1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2$$

$$= 1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2$$

Q14aiv Consider the size $2m$ group, $0 \le m \le n$.
Number of ways of selecting leaders first for each group is n^2.
Number of ways of selecting the remaining members is $\binom{n-1}{m-1}^2$.

.: total number of ways for $0 \le m \le n$ is

$$n^2\left[\binom{n-1}{0}^2 + \binom{n-1}{1}^2 + \cdots + \binom{n-1}{n-1}^2\right] = n^2\binom{2(n-1)}{(n-1)}.$$

Q14bi $\frac{\sin 3\theta}{4} = \frac{\sin(2\theta + \theta)}{4} = \frac{\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}{4}$

$$= \frac{2\sin\theta\cos^2\theta + \left(1 - 2\sin^2\theta\right)\sin\theta}{4} = \frac{3}{4}\sin\theta - \sin^3\theta$$

.: $\sin^3\theta - \frac{3}{4}\sin\theta + \frac{\sin 3\theta}{4} = 0$

Q14bii $x = 4\sin\theta$ and $x^3 - 12x + 8 = 0$

.: $64\sin^3\theta - 48\sin\theta + 8 = 0$, $\sin^3\theta - \frac{3}{4}\sin\theta + \frac{1}{8} = 0$.

Comparing with bi, $\frac{\sin 3\theta}{4} = \frac{1}{8}$.: $\sin 3\theta = \frac{1}{2}$

Q14biii $\sin^2\frac{\pi}{18} + \sin^2\frac{5\pi}{18} + \sin^2\frac{25\pi}{18}$

$$= \frac{1}{2}\left(1 - \cos\frac{\pi}{9}\right) + \frac{1}{2}\left(1 - \cos\frac{5\pi}{9}\right) + \frac{1}{2}\left(1 - \cos\frac{25\pi}{9}\right)$$

$$= \frac{1}{2}\left[3 - \left(\cos\frac{\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{25\pi}{9}\right)\right]$$

$$= \frac{1}{2}\left[3 - \left(\cos\frac{\pi}{9} + 2\cos\frac{5\pi}{3}\cos\frac{10\pi}{9}\right)\right]$$

$$= \frac{1}{2}\left[3 - \left(\cos\frac{\pi}{9} + \cos\frac{10\pi}{9}\right)\right] = \frac{1}{2}\left[3 - \left(\cos\frac{\pi}{9} - \cos\frac{\pi}{9}\right)\right] = \frac{3}{2}$$

Please inform mathline@itute.com re conceptual and/or mathematical errors.