2020 NSW ESA Mathematics Extension 2 Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
B	B	C	A	D	A	D	B	C	B

Q1 $\sqrt{(-1)^2+18^2+(-6)^2}=19$ B

Q2 $(z-(3+i))(z-(3-i))=z^2+pz+q$, $p=-6$, $q=10$ B

Q3 $x=1-2\lambda$, $y=3+4\lambda$, .: $y+2x=5$ C

Q4 $\frac{z^2}{|z|}=\frac{(r\text{cis}\theta)^2}{r}=r\text{cis}2\theta$ A

Q5 Let $x=A\cos nt$, $v=-An\sin nt$, $a=-An^2\cos nt$,
max $|v|=An=4$, max $|a|=An^2=6$, .: $n=\frac{3}{2}$, $T=\frac{2\pi}{n}=\frac{4\pi}{3}$ D

Q6 A

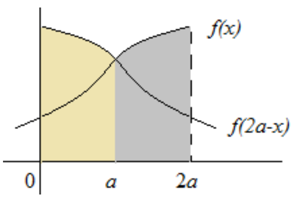
$$\int\frac{1}{x^2+4x+10}dx=\frac{1}{\sqrt{6}}\int\frac{\sqrt{6}}{(x+2)^2+(\sqrt{6})^2}dx=\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)+c$$

Q7 $(2^{11}-1)$ is not prime but 11 is prime. D

Q8 The given statement is true, its negation is false. B

Q9 Let $D=|z-2|+|z+2|$, $D^2=|z-2|^2+2|z-2||z+2|+|z+2|^2$
$D^2=10+2\sqrt{25-16\cos^2\theta}$, max $D^2=20$.: max $D=2\sqrt{5}$ C

Q10 $f(2a-x)$ is the translation $2a$ to the left and then reflection in the y-axis of $f(x)$.



$$\int_0^{2a}f(x)dx=\int_0^a f(x)dx+\int_a^{2a}f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$$
$$=\int_0^a f(x)+f(2a-x)dx$$ B

Section II

Q11ai $|-1+4i|=\sqrt{17}$

Q11aii $w\overline{z}=(-1+4i)(2+i)=-6+7i$

Q11b

$$\int_1^e x\ln x\,dx=\left[\frac{x^2}{2}\ln x\right]_1^e-\int_1^e\frac{x^2}{2}\frac{1}{x}dx=\left[\frac{x^2}{2}\ln x\right]_1^e-\left[\frac{x^2}{4}\right]_1^e=\frac{e^2+1}{4}$$

Q11c $x=0$, $v=1$, $a=v^2+v$, .: $a>0$ and $v\ge 1$

$v\frac{dv}{dx}=v^2+v$, $\frac{dv}{dx}=v+1$, $\frac{dx}{dv}=\frac{1}{v+1}$, $x=\ln(v+1)+c$, $c=-\ln 2$

.: $x=\ln\left(\frac{v+1}{2}\right)$

Q11d $(\tilde{u}-\tilde{v}).(\tilde{u}+\tilde{v})=\tilde{u}.\tilde{u}-\tilde{v}.\tilde{v}=0$, .: $14-p^2-5=0$, $p=\pm 3$

Q11e Let $z=a+bi$.

$z^2+3z+(3-i)=(a^2+3a+3-b^2)+(2ab+3b-1)i=0$

.: $a^2+3a+3-b^2=0$ and $2ab+3b-1=0$

.: $a^2+3a+3=b^2$ and $b=\frac{1}{2a+3}$

.: $a^2+3a+3=\frac{1}{4(a^2+3a+3)-3}$

.: $4(a^2+3a+3)^2-3(a^2+3a+3)-1=0$. Since $4(a^2+3a+3)+1\neq 0$

.: $(a^2+3a+3)-1=0$, .: $a=-1,-2$ and $b=1,-1$ respectively

.: $z=-1+i$ or $z=-2-i$

Q12ai $R+200\sin 30°-50g=0$, $R=400$

Q12aii $200\cos 30°-0.3\times 400\approx 53.2$ N

Q12aiii Constant $a\approx\frac{53.2}{50}$, $v=u+at\approx 0+\frac{53.2}{50}\times 3\approx 3.19$ m s^{-1}

Q12bi At $t=0$, $v_x=u\cos\theta$, $v_y=u\sin\theta$, $a_x=0$, $a_y=-g$, $(0,0)$

At time t , $x=\int_0^t v_x dt=ut\cos\theta$, $v_y=u\sin\theta+\int_0^t -g\,dt=u\sin\theta-gt$

and $y=\int_0^t(u\sin\theta-gt)dt=ut\sin\theta-\frac{1}{2}gt^2$

.: $\tilde{r}(t)=\begin{pmatrix}ut\cos\theta\\ ut\sin\theta-\frac{1}{2}gt^2\end{pmatrix}$

Q12bii Eliminate t from $x=ut\cos\theta$, $y=ut\sin\theta-\frac{1}{2}gt^2$

$y=x\frac{\sin\theta}{\cos\theta}-\frac{gx^2}{2u^2}\frac{1}{\cos^2\theta}=x\tan\theta-\frac{gx^2}{2u^2}(1+\tan^2\theta)$

$=-\frac{gx^2}{2u^2}\left(\tan^2\theta-\frac{2u^2}{gx}\tan\theta+1\right)$

Q12biii $x=R$, $y=-\frac{gR^2}{2u^2}\left(\tan^2\theta-\frac{2u^2}{gR}\tan\theta+1\right)=0$

$\tan\theta=\frac{u^2}{gR}\pm\sqrt{\frac{u^2}{gR}-1}$

Given $u^2>gR$, $\frac{u^2}{gR}>\frac{u^2}{gR}-1>0$.: two values of $\tan\theta$, $0<\theta<\frac{\pi}{2}$

Q13a $T=\frac{2\pi}{n}=\frac{\pi}{3}$.: $n=6$

$x-\sqrt{3}=2\sqrt{3}\cos 6t$.: $x=\sqrt{3}(2\cos 6t+1)$

Q13b $\tilde{r} = \begin{pmatrix} 3+\lambda_1 \\ -1+2\lambda_1 \\ 7+\lambda_1 \end{pmatrix}$, $\tilde{r} = \begin{pmatrix} 3-2\lambda_2 \\ -6+\lambda_2 \\ 2+3\lambda_2 \end{pmatrix}$

At intersection, components are equal, $\lambda_1 = -2$ and $\lambda_2 = 1$

Intersection $\begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

Q13ci $x = \sqrt{(a+b)^2 - (a-b)^2} = 2\sqrt{ab}$

Given $a > b \geq 0$, $a - b > 0$ (*problematic for the following*)

$\therefore\ a+b > x\ \therefore\ a+b > 2\sqrt{ab}$, $\frac{a+b}{2} > \sqrt{ab}$ (*instead of* $\frac{a+b}{2} \geq \sqrt{ab}$)

Q13cii $(p-2q)^2 \geq 0\ \therefore\ p^2 + 4q^2 \geq 4pq$

Q13di $z + \bar{z} = 2\,\text{Re}(z) = 2\cos n\theta$

Q13dii $(z+\bar{z})^4 = (2\cos\theta)^4 = 16\cos^4\theta$

$(z+\bar{z})^4 = z^4 + 4z^3\bar{z} + 6z^2\bar{z}^2 + 4z\bar{z}^3 + \bar{z}^4 = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore\ 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore\ \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Q13diii $\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{1}{8}\int_0^{\frac{\pi}{2}} (\cos 4\theta + 4\cos 2\theta + 3)\, d\theta$

$= \frac{1}{8}\left[\frac{\sin 4\theta}{4} + \frac{4\sin 2\theta}{2} + 3\theta\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{16}$

Q14ai $z_2 = e^{i\frac{\pi}{3}} z_1$, z_2 is the anticlockwise rotation of z_1 by $\frac{\pi}{3}$ about the origin in the Argand plane $\therefore \angle AOB = 60°$.

$|z_2| = \left|e^{i\frac{\pi}{3}} z_1\right| = \left|e^{i\frac{\pi}{3}}\right| |z_1| = |z_1|$, $\therefore\ \Delta OAB$ is equilateral.

Q14aii $z_1^{\ 2} + z_2^{\ 2} = z_1^{\ 2}\left(1 + e^{i\frac{2\pi}{3}}\right)$, $z_1 z_2 = z_1^{\ 2} e^{i\frac{\pi}{3}}$

$e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$1 + e^{i\frac{2\pi}{3}} = 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = 1 - \frac{1}{2} + \frac{\sqrt{3}}{2}i = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore\ z_1^{\ 2} + z_2^{\ 2} = z_1 z_2$

Q14b $t = 0$, $x = 0$, $v = 0$, $a = \frac{dv}{dt} = 10\left(1 - (0.01v)^2\right)$

$\frac{dt}{dv} = \frac{1}{10}\frac{1}{(1-0.01v)(1+0.01v)} = \frac{1}{20}\left(\frac{1}{1-0.01v} + \frac{1}{1+0.01v}\right)$

$20t = \int_0^v \left(\frac{1}{1-0.01v} + \frac{1}{1+0.01v}\right) dv = \frac{\ln(1-0.01v)}{-0.01} + \frac{\ln(1+0.01v)}{0.01}$

$\therefore\ 0.2t = \ln\left(\frac{1+0.01v}{1-0.01v}\right)$. When $t = 5$, $\frac{1+0.01v}{1-0.01v} = e\ \therefore\ v = \frac{100(e-1)}{e+1}$

Q14c $n = 2$, $\frac{1}{2^2} < \frac{2-1}{2}$ is true

Assume it is true for $n = k$, $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$

For $n = k+1$, $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \frac{k-1}{k} + \frac{1}{(k+1)^2}$

$\frac{k-1}{k} + \frac{1}{(k+1)^2} = \frac{(k-1)(k+1)^2 + k}{k(k+1)} = \frac{k^3 + k^2 - 1}{k(k+1)^2} < \frac{k^3 + k^2}{k(k+1)^2}$

$= \frac{k}{k+1} = \frac{(k+1)-1}{(k+1)}$

$\therefore\ \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \frac{(k+1)-1}{(k+1)}$ is true.

Hence $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}$ is true for all $n \geq 2$.

Q14d If $\log_n(n+1)$ is rational for $n > 1$, then $\log_n(n+1) = \frac{p}{q}$ where p, q are natural numbers.

$(n+1) = n^{\frac{p}{q}}\ \therefore\ (n+1)^q = n^p$. The last statement implies the impossibility, even number = odd number .

$\therefore\ \log_n(n+1)$ is irrational.

Q15ai If $k+1$ is divisible by 3, $k^3 + 1 = (k+1)(k^2 - k + 1)$ is also divisible by 3.

Q15aii Contrapositive: If $k^3 + 1$ is not divisible by 3, then $k+1$ is not divisible by 3

Q15aiii The converse "If $k^3 + 1$ is divisible by 3, then $k+1$ is divisible by 3" is true.

Proof: If $k^3 + 1 = (k+1)(k^2 - k + 1)$ is divisible by 3, either $k+1$ or $k^2 - k + 1$ or both are divisible by 3.

Consider $k^2 - k + 1 = 3m$ where m is a positive integer.

$k^2 - k + 1 - 3m = 0$, $k = \frac{1 \pm \sqrt{1 - 4(1-3m)}}{2} = \frac{1 \pm \sqrt{3(4m-1)}}{2}$

$\therefore\ k+1 = \frac{3 \pm \sqrt{3(4m-1)}}{2}$ is divisible by 3 when $4m - 1 = 3p^2$ and p is odd, as shown below.

$k+1 = \frac{3 \pm \sqrt{3(3p^2)}}{2} = \frac{3 \pm 3p}{2} = \frac{3(1 \pm p)}{2} = 3q$, q an integer.

Q15bi $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \tilde{b} - \tilde{a}$, $\overrightarrow{AC} = \frac{n}{m+n}\overrightarrow{AB} = \frac{n}{m+n}\left(\tilde{b} - \tilde{a}\right)$

Q15bii $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \tilde{a} + \frac{n}{m+n}\left(\tilde{b} - \tilde{a}\right) = \frac{m}{m+n}\tilde{a} + \frac{n}{m+n}\tilde{b}$

Q15biii $\overrightarrow{OS} = \tilde{r} + \frac{1}{2}\tilde{p}$, $\overrightarrow{OT} = \frac{m}{m+n}\tilde{r} + \frac{n}{m+n}\tilde{p}$

$\overrightarrow{OS} \,//\, \overrightarrow{OT}$, $\frac{m}{m+n} = 2\left(\frac{n}{m+n}\right)$, $m = 2n\ \therefore\ \overrightarrow{OT} = \frac{2}{3}\tilde{r} + \frac{1}{3}\tilde{p}$

Q15biv $m = 2n$, $\frac{PT}{TR} = \frac{m}{n} = \frac{2}{1}$, $PT : TR = 2:1$

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Q16ai $a=\frac{\text{net force}}{\text{mass}}=\frac{4mg-2mg-2kv}{6m}=\frac{mg-kv}{3m}$

Q16aii $t=0$, $v=0$, $a=\frac{dv}{dt}=\frac{mg-kv}{3m}$, $\frac{dt}{dv}=\frac{3m}{mg-kv}$

$t=\frac{3m}{k}\int_0^v \frac{1}{\frac{mg}{k}-v}dv$, $\frac{kt}{3m}=\ln\frac{\frac{mg}{k}}{\frac{mg}{k}-v}$

When $t=\frac{3m}{k}\ln 2$, $\ln 2=\ln\frac{\frac{mg}{k}}{\frac{mg}{k}-v}$ $\therefore$ $v=\frac{mg}{2k}$

Q16bi $I_n=\int_0^{\frac{\pi}{2}}\sin^{2n}2\theta\sin 2\theta\,d\theta=\int_0^{\frac{\pi}{2}}\sin^{2n}2\theta\frac{d}{d\theta}\left(\frac{-\cos 2\theta}{2}\right)d\theta$

$=\left[\sin^{2n}2\theta\left(\frac{-\cos 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\left(\frac{-\cos 2\theta}{2}\right)\left(\frac{d}{d\theta}\sin^{2n}2\theta\right)d\theta$

$=0+\int_0^{\frac{\pi}{2}}2n\cos^2 2\theta\sin^{2n-1}2\theta\,d\theta=2n\int_0^{\frac{\pi}{2}}\left(1-\sin^2 2\theta\right)\sin^{2n-1}2\theta\,d\theta$

$=2n\left(\int_0^{\frac{\pi}{2}}\sin^{2n-1}2\theta\,d\theta-\int_0^{\frac{\pi}{2}}\sin^{2n+1}2\theta\,d\theta\right)$

$\therefore$ $I_n=2n\left(I_{n-1}-I_n\right)$ $\therefore$ $I_n=\frac{2n}{2n+1}I_{n-1}$ for $n\ge 1$

Q16bii $I_n=\frac{2n}{2n+1}I_{n-1}=\frac{2n}{2n+1}.\frac{2(n-1)}{2n-1}.\frac{2(n-2)}{2n-3}I_{n-3}\cdots$

$=\frac{2^n n!}{(2n+1)(2n-1)(2n-3)\cdots(3)(1)}I_0$

$=\frac{2^n n!2n(2n-2)(2n-4)\cdots(4)(2)}{(2n+1)(2n-1)(2n-3)\cdots(3)(1)2n(2n-2)(2n-4)\cdots(4)(2)}I_0$

$=\frac{2^n 2^n n!n(n-1)(n-2)\cdots(2)(1)}{(2n+1)(2n)(2n-1)(2n-2)\cdots(2)(1)}I_0$

$=\frac{2^{2n}(n!)^2}{(2n+1)!}$ Note : $I_0=\int_0^{\frac{\pi}{2}}\sin 2\theta\,d\theta=-\left[\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}=1$

Q16biii Given $J_n=\int_0^1 x^n\left(1-x^n\right)dx$ and let $x=\sin^2\theta$

$J_n=\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\left(1-\sin^{2n}\theta\right)\left(2\sin\theta\cos\theta\right)d\theta$

$=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}2^{2n}\sin^{2n}\theta\cos^{2n}\theta\left(2\sin\theta\cos\theta\right)d\theta$

$=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}\left(2\sin\theta\cos\theta\right)^{2n+1}d\theta=\frac{1}{2^n}\int_0^{\frac{\pi}{2}}\sin^{2n+1}2\theta\,d\theta$

$=\frac{1}{2^{2n}}.\frac{2^{2n}(n!)^2}{(2n+1)!}=\frac{(n!)^2}{(2n+1)!}$

Q16biv Since $I_n=\frac{2n}{2n+1}I_{n-1}$ for $n\ge 1$

$\therefore$ $I_n<I_{n-1}<I_{n-2}<\cdots<I_0=1$ $\therefore$ $\frac{2^{2n}(n!)^2}{(2n+1)!}<1$ for $n\ge 1$ or

$\frac{2^{2n}(n!)^2}{(2n+1)!}\le 1$ for $n\ge 0$. Hence $\left(2^n n!\right)^2\le(2n+1)!$

Please inform mathline@itute.com re conceptual and/or mathematical errors.

