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2020 Specialist Mathematics

Year 12 Application Task

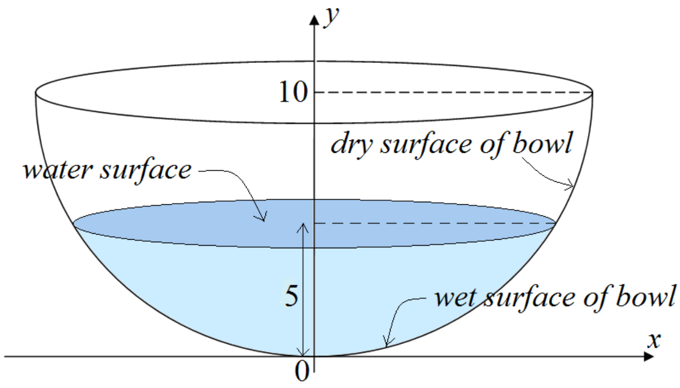
(Time allowed: 4 hours plus)

Theme: Volume and quality of water in a lake

The volume of water in a freshwater lake changes due to rainfall, rain water runoff from the surrounding land, water seepage/leakage through the lake-bed, evaporation, and pumping for irrigation. The lake is contaminated by insecticides and fertilizers run off from the surrounding farms into the streams feeding the lake.

Part I (80-90 min)

The following diagram shows a **hemispherical** bowl of radius 10 units in its upright position.
The thickness of the wall is negligible. Axes $(x \text{ and } y)$ are added to the diagram.



The bowl is filled with water to a depth of 5 units.

a. Calculate the area of the circular water surface.

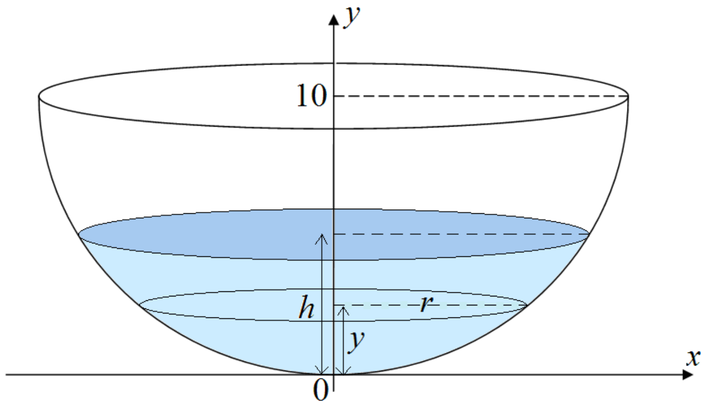
b. Find the equation of the semicircle representing the upright profile of the bowl.
State the domain and range of the semicircle.

c. Write a definite integral for the volume of water in the bowl.

d. Calculate the volume of water in the bowl.

The area of the wet surface of the bowl can be calculated by the following definite integral.

$\text{Area} = \int_0^h 2\pi r \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$, where h units is the depth of water in the bowl, and $2\pi r$ is the horizontal circumference of the bowl at a depth of y for $0 \le y \le h$.

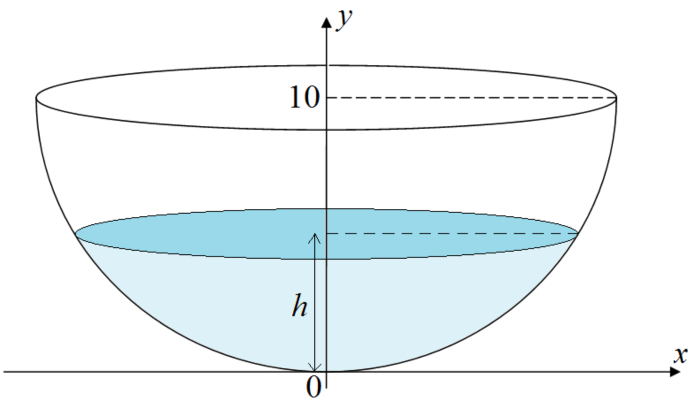


e. Show that $r = \sqrt{20y - y^2}$.

f. Hence show that $1+\left(\frac{dx}{dy}\right)^2 = \frac{100}{r^2}$.

g. Calculate the area of wet surface of the bowl when it is filled to a depth of 5 units, i.e. $h = 5$.

Let h units be the depth of water in the bowl. The radius of the hemispherical bowl is 10 units.



h. Determine a formula for the area of the water surface in terms of h.

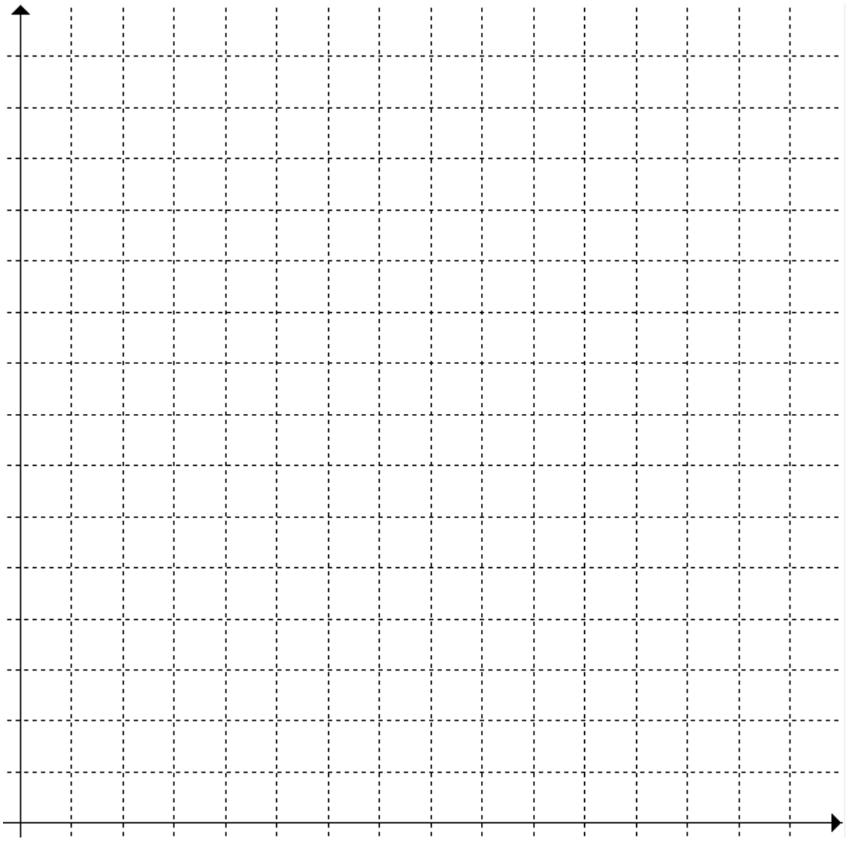
i. Write a definite integral for finding the volume of water in the bowl.
Determine a formula for the volume of water in terms of h.

j. Write a definite integral for the area of the wet surface of the bowl.
Show that the area of the wet surface of the bowl is given by $20\pi h$ square units.

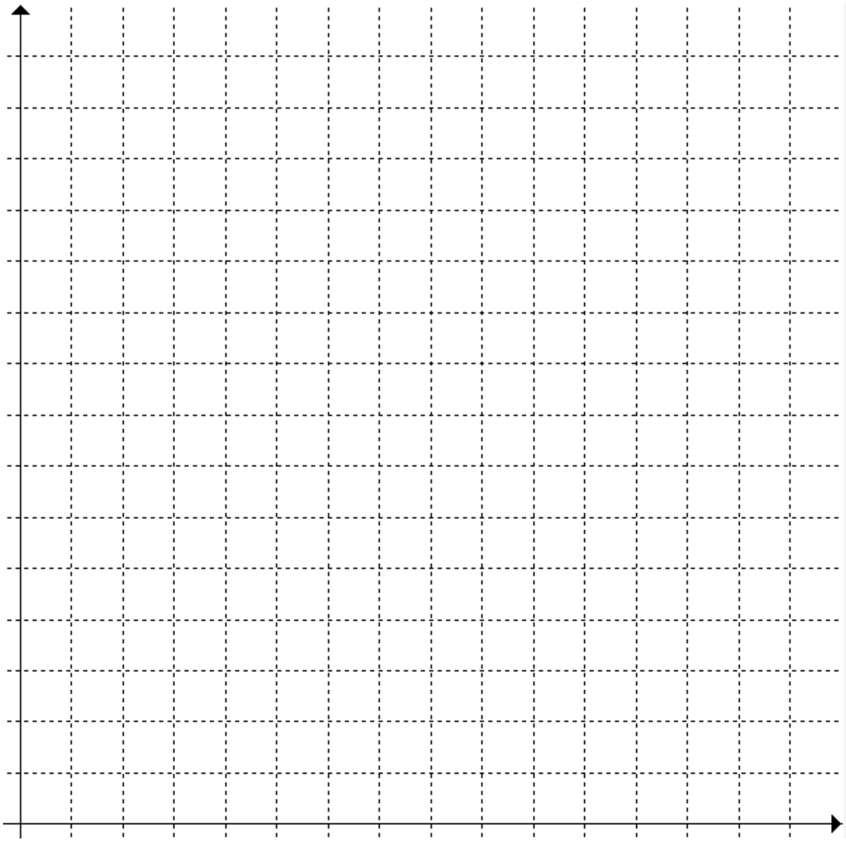
k. Sketch graphs showing

(i) the relationship between area of water surface in the bowl and depth,
(ii) the relationship between volume of water in the bowl and depth, and
(iii) the relationship between wet surface area of the bowl and depth.
Comment on the shape/features/trend of each graph.

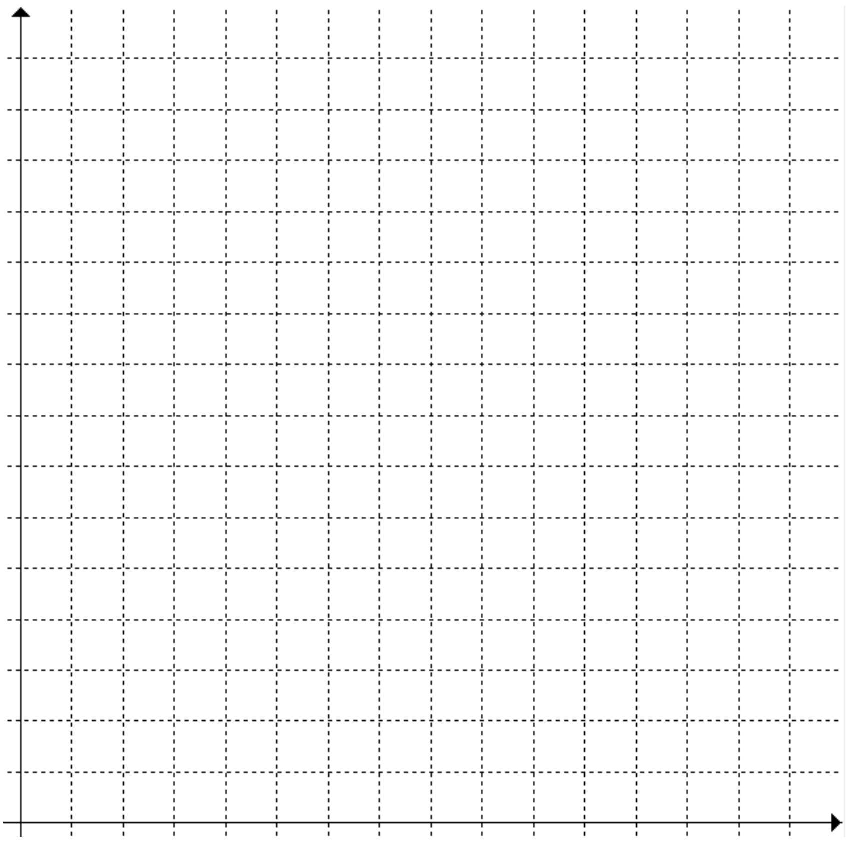
(i)



(ii)



(iii)



Let the radius of the **water surface** be 5 units when the bowl is filled with water to certain depth for the following parts l, m and n.

l. Determine the depth of water in the bowl.

m. Determine the volume of water in the bowl.

n. Determine the area of the wet surface of the bowl.

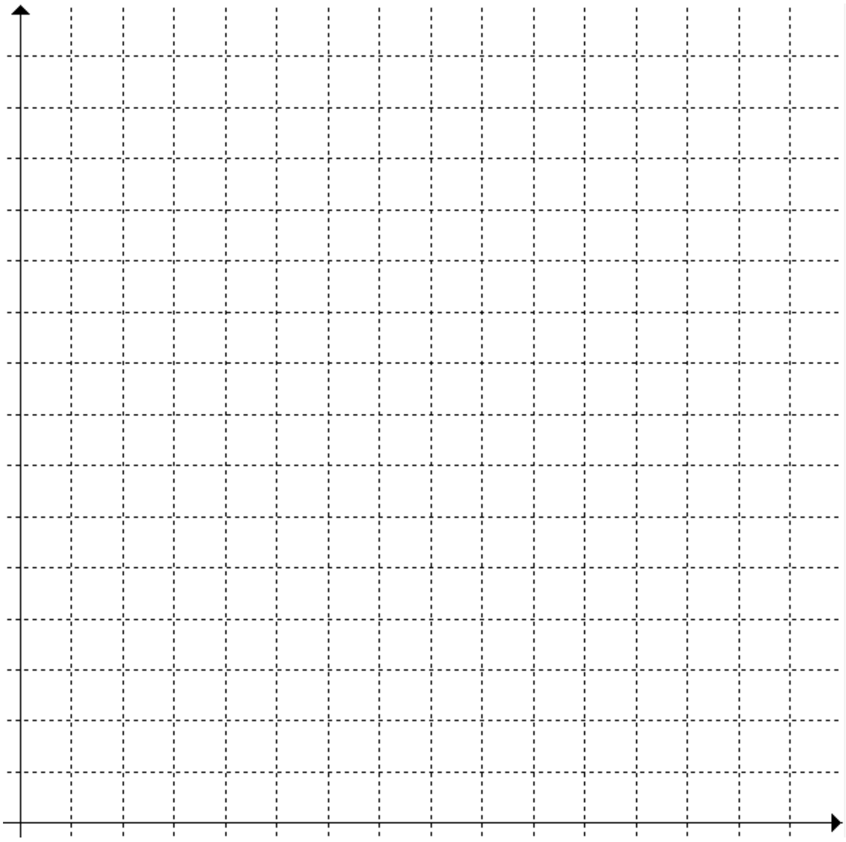
Let R units be the radius of the hemispherical bowl and h units is the depth of water in the bowl.

o. Investigate the effects of changing R on each of the graphs in part k.

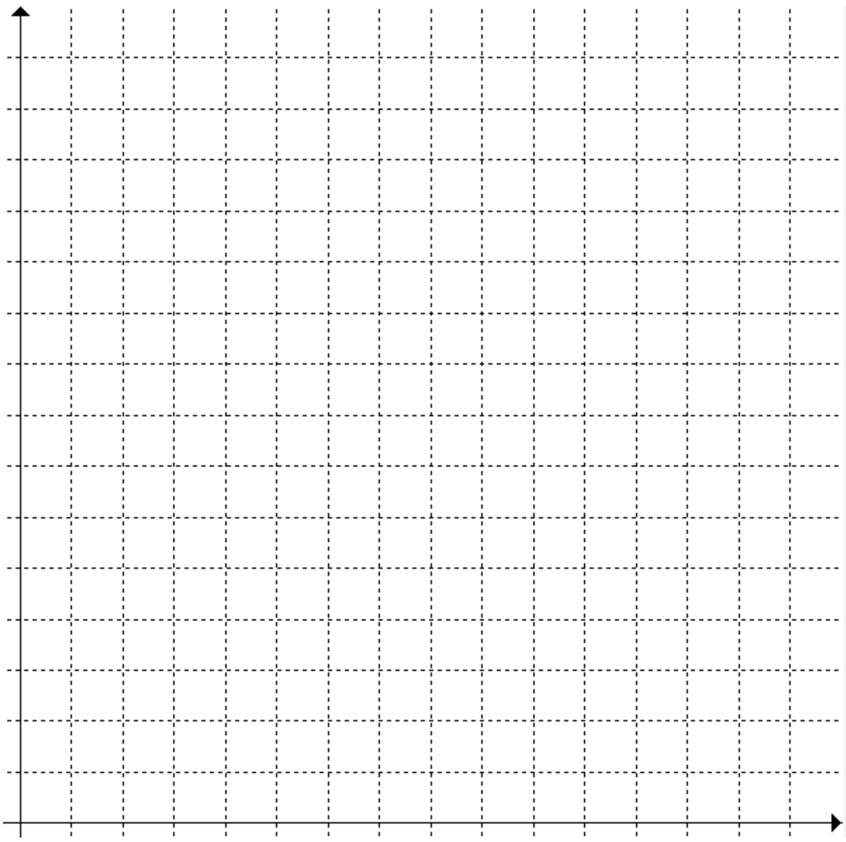
Suggestion: In each case sketch further graphs on the same set of axes using at least three more suitable R values for $10 < R < 500$.

Discuss/comment on your findings.

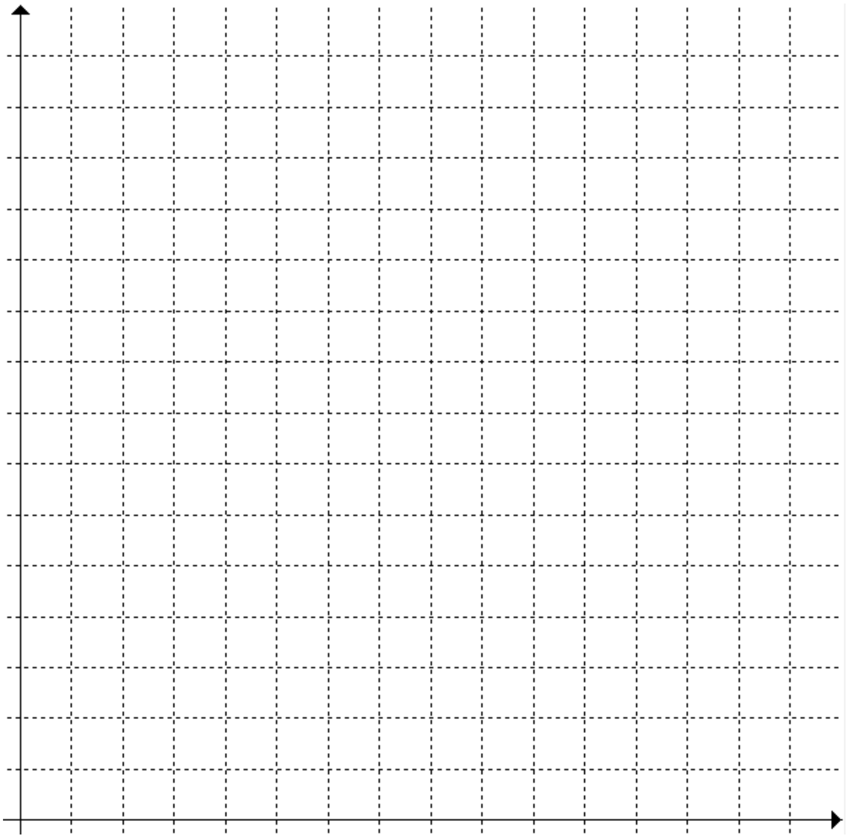
(i)



(ii)



(iii)



p. In terms of R and h determine a formula for each of the following quantities.

The area of the water surface in the bowl

The volume of water in the bowl

The area of the wet surface of the bowl

Theme: Volume and quality of water in a lake

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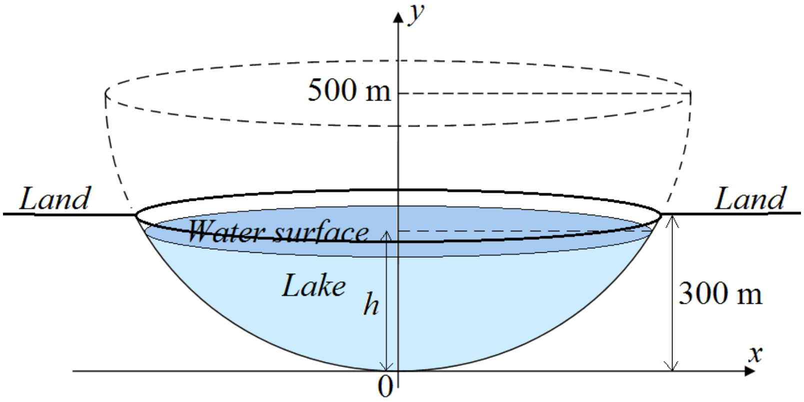
Part II (80-90 min)

The diagram below shows a hemisphere of radius 500 m. The hemisphere is partly dotted.
The **circular freshwater lake** is represented by the lower section of the hemisphere.
The surrounding land will start to flood if the lake is filled to capacity (maximum depth of 300 m).



** Water seeps through the lake-bed at a constant rate of 0.0002 m^3 per day for each m^2 of the lake-bed.

** Evaporation of water in the lake occurs continuously. It is measured in m^3 per day for each m^2 of water surface of the lake. It changes due to variations in air temperature and wind speed.
At time t over a 60-day period it is given by

$$\text{rate of evaporation P} = \frac{0.0001}{1+0.0006(t-30)^2} \text{ for } 0 \le t \le 60.$$

a. Determine the **average** rate of evaporation of water (in m^3 per day) for each m^2 of water surface of the lake over a 60-day period.

b. **Without using CAS** determine the maximum and minimum rates of evaporation in the 60-day period. Marks are given for reasoning/explanation only.

Let $h = 295\,\text{m}$ at a particular time in the 60-day period.

c. Show that the area of the circular water surface of the lake is $207975\,\pi\,\text{m}^2$ at that time. Hence calculate the rate of evaporation in m^3 per day.

d. Show that the area of the lake-bed covered by water is $295000\,\pi\,\text{m}^2$ at that time. Hence calculate the rate of water seepage through the lake-bed in m^3 per day.

In the 60-day period, besides evaporation and seepage, the volume of water in the lake also changes because of rainfall, rain water runoff and pumping for irrigation. Assume that water is added to the lake due to rainfall and rain water runoff from the surrounding land at a constant rate of $115\,000\,\text{m}^3$ per day, and water from the lake is pumped out for irrigation at a constant rate of $49\,300\,\text{m}^3$ per day.

e. Determine the rate of change of the volume of water in the lake (in m^3 per day) due to the combined effects of rainfall, rain water runoff and pumping for irrigation.

At the start of the 60-day period, i.e. $t = 0$, the depth of the lake is $h = 294$ m.

f. Discuss whether the land surrounding the lake will be flooded in the 60-day period.
Full explanation with calculations is required, considering all changes that could affect the volume of water in the lake.

Extra space for part f

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Part III (80-90 min)

Assume the following information/data.

** Evaporation and water seepage/leakage through the lake-bed are insignificant.

** The volume of water in the lake is 1.0916×10^{8} m^3 at the start of the 60-day period, i.e. when $t=0$.

** The concentration of contaminants in the lake is 0.001850 kg per m^3 when $t=0$.

Write values/coefficients with appropriate number of decimal places to minimise the effects on your solutions due to rounding errors.

a. Further assumptions for this part only.

** In the 60-day period water is added to the lake due to rainfall and rain water runoff from the surrounding land at a constant rate of $115\,000$ m^3 per day, and no water is pumped out from the lake in the first 10 days.

** Water added to the lake (due to rainfall and rain water runoff from the surrounding land) contains 0.001500 kg per m^3 of contaminants.

Let Q kg be the amount of contaminants in the lake at time t.

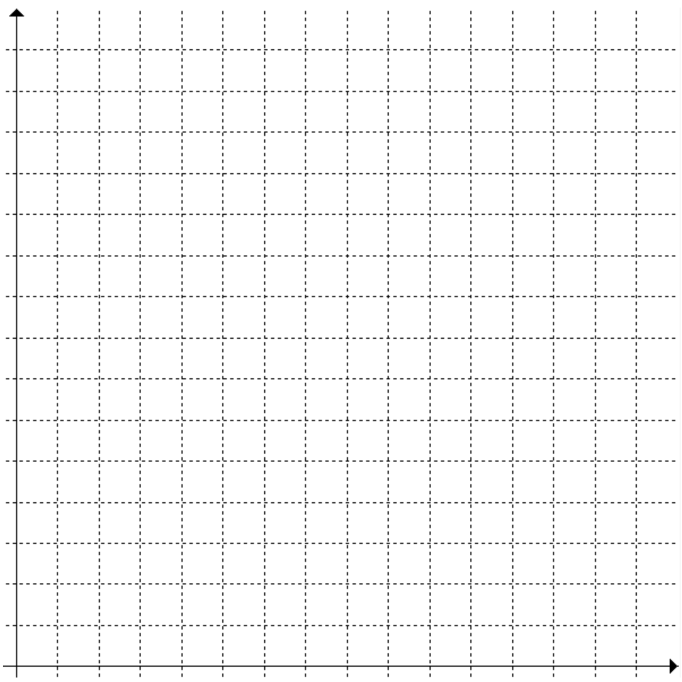
(i) Find the concentration of contaminants in the lake at the end of day 10.

(ii) Write an expression for the concentration of contaminants in the lake at time t where $0 \le t \le 10$.

(iii) Write a differential equation which describes the rate of change (kg per day) of the amount of contaminants Q in the lake for $0 \le t \le 10$.

(iv) **Without using CAS** find $Q(t)$ where $0 \le t \le 10$.

(v) Sketch the graph of $Q(t)$ for $0 \le t \le 10$.



b. Further assumptions for this part only. **This part is not related to part a.**

** In the 60-day period water is added to the lake due to rainfall and rain water runoff from the surrounding land at a constant rate of 115000 m^3 per day, and water is pumped out from the lake for irrigation at the same rate of 115000 m^3 per day.

** Water added to the lake (due to rainfall and rain water runoff from the surrounding land) contains 0.001500 kg per m^3 of contaminants.

Let Q kg be the amount of contaminants in the lake at time t.

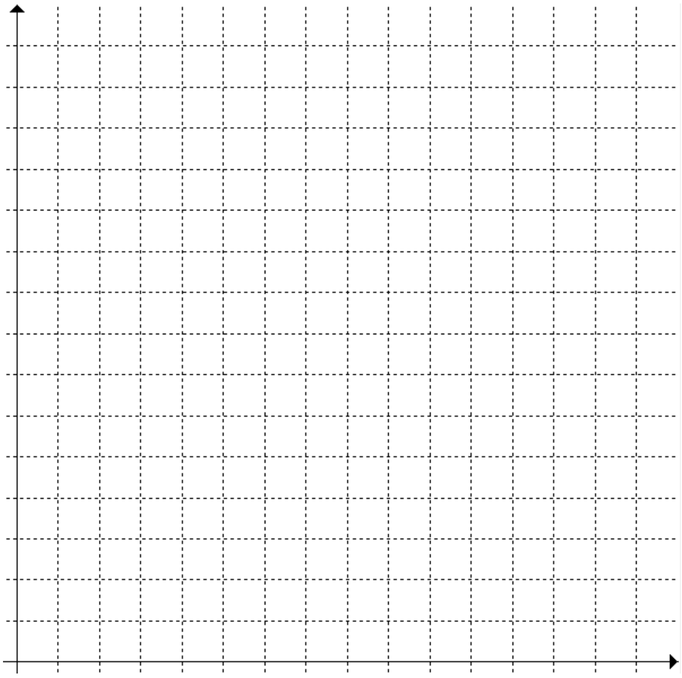
(i) Write an expression for the concentration of contaminants in the lake at time t where $0 \le t \le 60$.

(ii) Calculate the amount of contaminants added to the lake each day due to rainfall and rain water runoff from the surrounding land.

(iii) Write a differential equation which describes the rate of change (kg per day) of the amount of contaminants Q in the lake for $0 \le t \le 60$.

(iv) **Without using CAS** find $Q(t)$ where $0 \le t \le 60$. Show working.

(iv) Sketch the graph of $Q(t)$ for $0 \le t \le 60$.



c. Further assumptions for this part only. **This part is not related to parts a and b.**

** In the 60-day period water is added to the lake due to rainfall and rain water runoff from the surrounding land at a constant rate of $115\,000$ m^3 per day, and water is pumped out from the lake for irrigation at a constant rate of $49\,300$ m^3 per day.

** Water added to the lake (due to rainfall and rain water runoff from the surrounding land) contains 0.001500 kg per m^3 of contaminants.

Let Q kg *be the amount of contaminants in the lake at time* t.

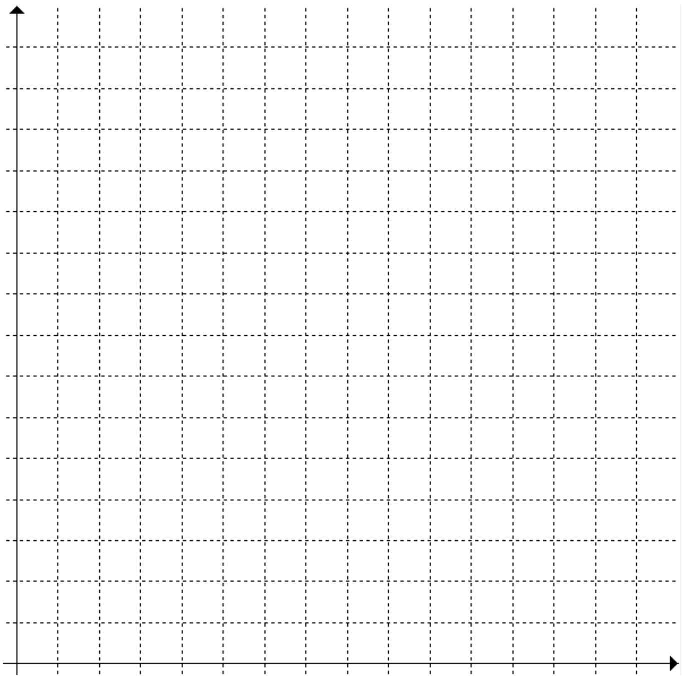
(i) In terms of Q write an expression for the concentration of contaminants (kg per m^3) in the lake at time t.

(ii) In terms of Q write an expression for the amount of contaminants removed from the lake each day due to pumping for irrigation at time t.

(iii) Set up a differential equation which describes the rate of change (kg per day) of the amount of contaminants Q in the lake.

(iv) By CAS find $Q(t)$ where $0 \le t \le 60$.

(v) Sketch the graph of $Q(t)$ for $0 \le t \le 60$.



(vi) Find the change in the total amount of contaminants in the lake in the 60-day period.

(vii) Explain why the graph in part (vi) is approximately linear.

End of Task