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2020 Specialist Mathematics

Year 12 Modeling or Problem Solving Task

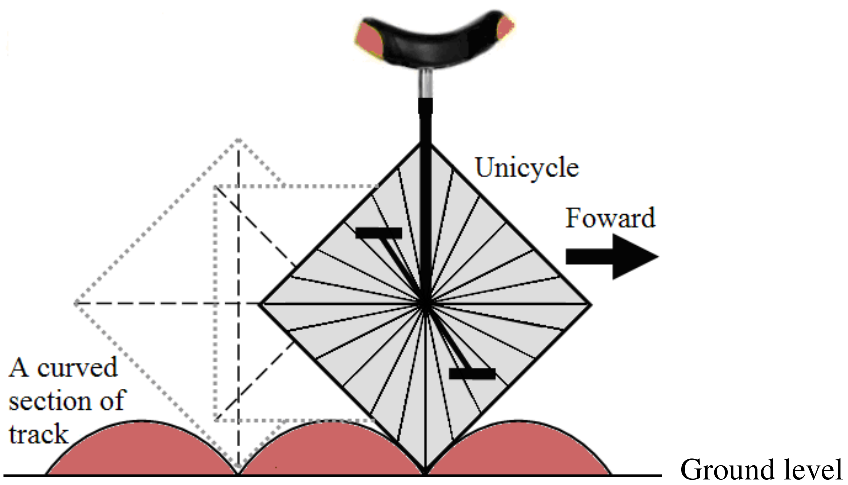
(Time allowed: 2.5 hours plus)

Modeling/Problem Solving Task

In this task length is measured in certain unit (not specified) and angle is in radian.
You do not need to state units in your answers.
Correct approximate values to 4 decimal places.

Theme: Unicycle with square wheel

A student built a unicycle with square wheel just for fun.
The **diagonal** of the square wheel is 2 in length.
He encountered some problems when he tried to ride the unicycle on a horizontal flat track.
His legs were not strong and powerful enough to make the square wheel to turn.
It would be a very bumpy ride even if he could turn the wheel.
He tried to solve the problems by making a track consisting of identical curved sections as shown below.



Three necessary requirements for the curved sections are specified below.

(1) Two adjacent sections make an angle of $\frac{\pi}{2}$ at the joining point.

(2) The length of each arc in a section equals the side length of the square wheel, i.e. $\sqrt{2}$.

(3) The highest point of a section must be $\frac{2-\sqrt{2}}{2}$ above the ground level. This is a necessary condition for the centre of the square wheel to remain at the same level of 1 above the ground level when the wheel turns and runs on each identical curved section.

Part I (80 – 90 minutes)

a. Show that the side length of the square wheel is $\sqrt{2}$, and the highest point of a section is $\frac{2-\sqrt{2}}{2}$ above the ground level.

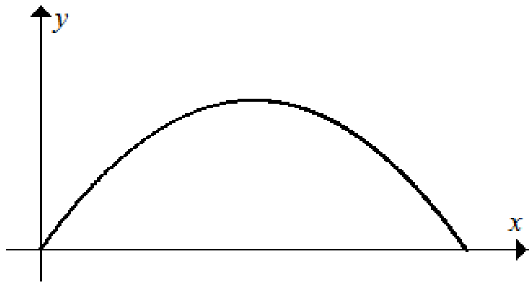
The student experimented with six tracks, 3 tracks consisting of sinusoidal sections and 3 tracks consisting of parabolic sections.

He designed each track using any 2 of the 3 requirements stated before, but each design did not meet the remaining one of the 3 necessary requirements.

Your task in Part I is to go over each track designed by the student and show that it does not meet the remaining requirement.

Three tracks consisting of sinusoidal sections (in questions b, c and d)

A sinusoidal section is shown in the following graph.



The Cartesian equation of the sinusoidal section is $y = a\sin(bx)$, where a and b are real constants to be determined.

b i. For a track to meet requirements (1) and (3), show/explain that $a = \dfrac{2-\sqrt{2}}{2}$ and $b = 2+\sqrt{2}$.

b ii. Write a definite integral for finding the length of the sinusoidal section (arc length).

b iii. Hence show that requirement (2) is not met.

c i For a track to meet requirements (1) and (2), show that $a \approx 0.3702$ and $b \approx 2.7014$.

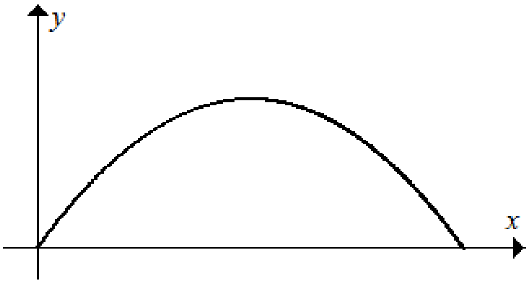
c ii Hence calculate the length of a sinusoidal section in this track to show that requirement (3) is not met.

d i For a track to meet requirements (2) and (3), show that $a \approx 0.2929$ and $b \approx 2.5974$.

d ii Hence determine the angle between any two adjacent sinusoidal sections at the joining point in this track to show that requirement (1) is not met.

Parabolic sections (in questions e, f and g)

A parabolic section is shown in the following graph.



The Cartesian equation of the parabolic section is $y = ax(x-b)$, where a and b are real constants to be determined.

e i. For a track to meet requirements (1) and (3), show that $a = -\frac{2+\sqrt{2}}{4}$ and $b = 2(2-\sqrt{2})$.

e ii. Write a definite integral for finding the length of the parabolic section.

e iii. Hence show that requirement (2) is not met.

f i For a track to meet requirements (1) and (2), show that $a \approx -0.8116$ and $b \approx 1.2321$.

f ii Hence calculate the length of a parabolic section in this track to show that requirement (3) is not met.

g i For a track to meet requirements (2) and (3), show that $a \approx -0.7499$ and $b \approx 1.2499$.

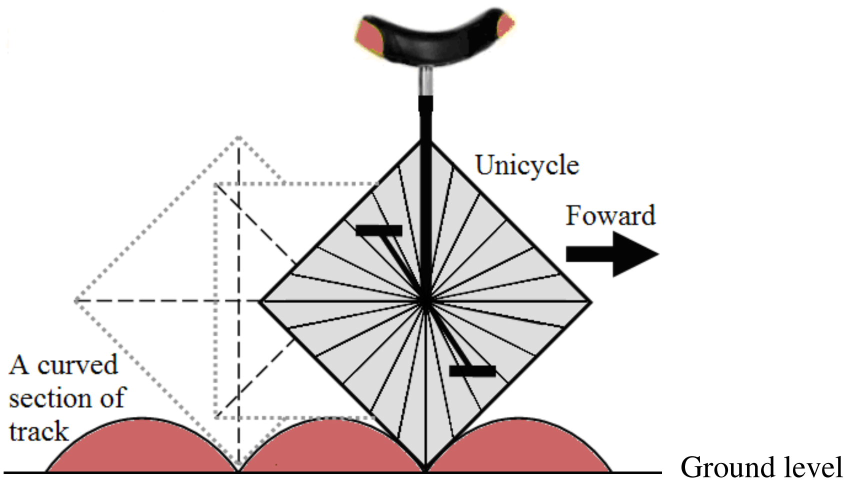
g ii Hence determine the angle between any two adjacent parabolic sections at the joining point in this track to show that requirement (1) is not met.

Modeling/Problem Solving Task (reprinted for Part II)

In this task length is measured in certain unit (not specified) and angle is in radian.
You do not need to state units in your answers.
Correct approximate values to 4 decimal places.

Theme: Unicycle with square wheel

A student built a unicycle with square wheel just for fun.
The **diagonal** of the square wheel is 2 in length.
He encountered some problems when he tried to ride the unicycle on a horizontal flat track.
His legs were not strong and powerful enough to make the square wheel to turn.
It would be a very bumpy ride even if he could turn the wheel.
He tried to solve the problems by making a track consisting of identical curved sections as shown below.



Three necessary requirements for the curved sections are specified below.

(1) Two adjacent sections make an angle of $\frac{\pi}{2}$ at the joining point.

(2) The length of each arc in a section equals the side length of the square wheel, i.e. $\sqrt{2}$.

(3) The highest point of a section must be $\frac{2-\sqrt{2}}{2}$ above the ground level. This is a necessary condition for the centre of the square wheel to remain at the same level of 1 above the ground level when the wheel turns and runs on each identical curved section.

In **Part I** you investigated tracks consisting of identical sinusoidal sections or parabolic sections.
None of them met the requirements specified above.

Part II (80 – 90 minutes)

In **Part II** your task is to find a suitable track consisting of identical curved sections for the **unicycle with square wheel**.
The track must meet all three necessary requirements.

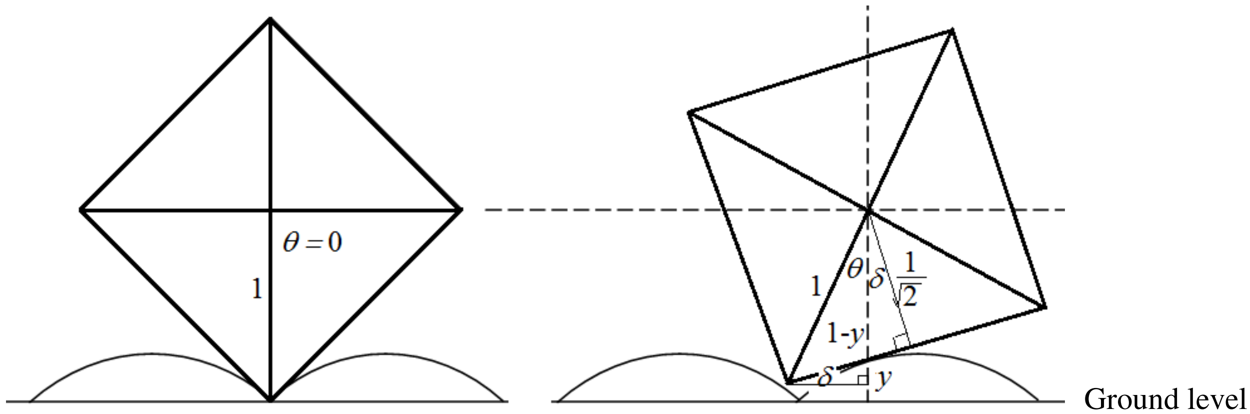
Parametric equations are used in **Part II**.

The parameter θ is the angle of rotation of the square wheel, where $0 \le \theta \le \frac{\pi}{2}$.

When the diagonal of the square wheel is vertical as shown below, $\theta = 0$, $x = 0$ and $y = 0$.

After a rotation of θ the square wheel touches the curved section at (x, y), the point of contact.

The distance from the wheel centre to the point of contact is $1 - y$.



The diagram on the right shows the orientation of the square wheel after a rotation of θ.

$\theta + \delta = \frac{\pi}{4}$, $\therefore\ \delta = \frac{\pi}{4} - \theta$

a. Find $\cos(\delta)$ in terms of y. Hence show that $y = 1 - \dfrac{1}{\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right)}$.

b. The side of the square wheel in contact with the curved section makes an angle δ with the horizontal.

Show that $\dfrac{dy}{dx} = \tan\left(\dfrac{\pi}{4} - \theta\right)$ at (x, y).

c. Use $\frac{dy}{d\theta}=\frac{dy}{dx}\times\frac{dx}{d\theta}$ to show $\frac{dx}{d\theta}=\frac{1}{\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)}$.

d. Use CAS to find $\int\frac{1}{\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)}\,d\theta$.

e. Hence show that $x=\frac{1}{\sqrt{2}}\log_e\frac{\cos\left(\frac{\pi}{4}-\theta\right)}{1+\sin\left(\frac{\pi}{4}-\theta\right)}+c$

f. Given $x=0$ when $\theta=0$, show that $x=\frac{1}{\sqrt{2}}\log_e\frac{\left(\sqrt{2}+1\right)\cos\left(\frac{\pi}{4}-\theta\right)}{1+\sin\left(\frac{\pi}{4}-\theta\right)}$.

g. Write a definite integral for finding the length of a curved section.
Evaluate the definite integral by CAS, and express your answer in exact form.

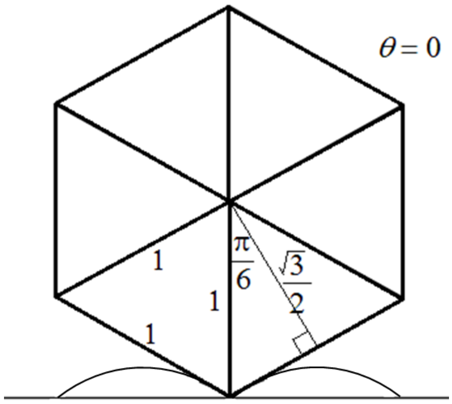
h. Verify that the angle between any two adjacent sections at the joining point is indeed $\frac{\pi}{2}$.

i. Verify that the distance of the highest point of a section from the ground level $y = 0$ is indeed $\frac{2-\sqrt{2}}{2}$.

j. Find the exact horizontal span of a section.

Now consider finding a track for a unicycle with **regular hexagonal wheel** of diagonal of 2 in length.

The parameter θ is the angle of rotation of the regular hexagonal wheel, where $0 \le \theta \le \frac{\pi}{3}$.
When the diagonal of the regular hexagonal wheel is vertical as shown below, $\theta = 0$, $x = 0$ and $y = 0$.
After a rotation of θ the regular hexagonal wheel touches the curved section at (x, y), the point of contact.
The distance from the wheel centre to the point of contact is $1 - y$.
Some measurements are shown on the diagram.



k. By following the same steps in finding the parametric equations of a section for a unicycle with square wheel, show that the parametric equations of a section for a unicycle with regular hexagonal wheel of diagonal of 2 in length are

$$y = 1 - \frac{\sqrt{3}}{2\cos\left(\frac{\pi}{6} - \theta\right)} \text{ and } x = \frac{\sqrt{3}}{2}\log_e \frac{\sqrt{3}\cos\left(\frac{\pi}{6} - \theta\right)}{1 + \sin\left(\frac{\pi}{6} - \theta\right)}$$

l. Use the parametric equations to verify that the following three necessary requirements for the curved sections are met.

(1) Two adjacent sections make an angle of $\frac{2\pi}{3}$ at the joining point.

(2) The length of each arc in a section equals the side length of the regular hexagonal wheel, i.e. 1.

(3) The highest point of a section is $\frac{2-\sqrt{3}}{2}$ above the ground level.

m. Find the parametric equations (of parameter θ) for a curved section starting from $(0,0)$ of a track suitable for a unicycle with a regular n-side polygonal wheel which has a diagonal of 2 in length.

n. Hence show that the span of a section $\to 0^+$ and $y \to 0^+$ as $n \to \infty$. Interpret this result/situation.

End of Task