2020 Specialist Mathematics

Year 12 Modelling Task

(Time allowed: 2.0 hours plus)

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Modelling Task

Theme: Spread of virus in a population

Assumed knowledge:
Functions, graphs, differential and integral calculus, differential equations and use of CAS

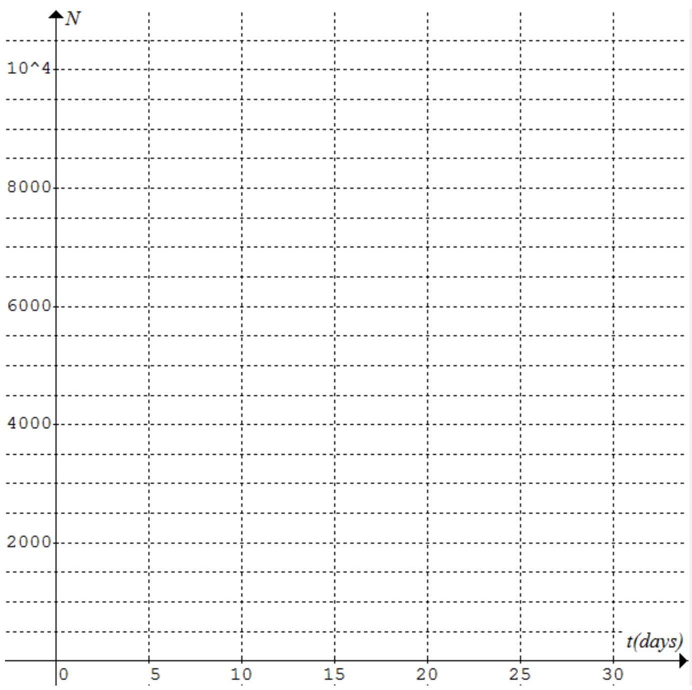
Part I (60 – 85 minutes)

Scenario 1
Specifications:
The population is infinite.
The spread of virus is NOT controlled.
The rate of growth of the number of people infected with the virus is proportional to the number of infected people N. The rate of growth is measured in number of people per day.

a. Scenario 1 can be modelled by differential equation $\frac{dN}{dt} = kN$ where k is a constant and its value can be adjusted to suit a given situation.
Use the method of separation of variables to show the solution to the differential equation is $N = N_0 e^{kt}$ where N_0 is the number of infected people initially (i.e. when $t = 0$).

b. Calculate the value of k (to 6 decimal places) if the number of infected people is doubled in 3 days.

c. Let $N_0 = 10$ and $k = 0.23$.
Sketch the graph of $N = N_0 e^{kt}$, $0 \le t \le 30$. Label the end points with their coordinates.



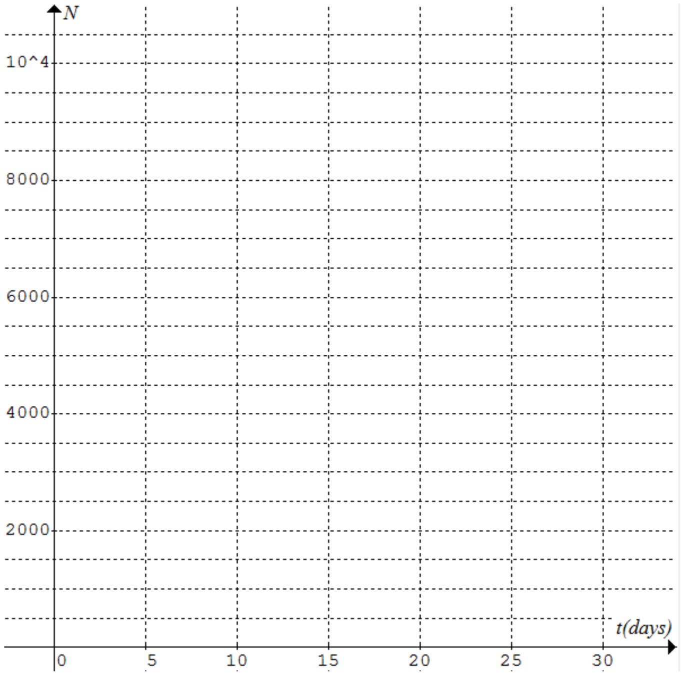
d. Investigate the effects of changing the value of k on the graph of $N = 10e^{kt}$.
Hint: Sketch the graphs of $N = 10e^{kt}$ for $k = 0.11, 0.14, 0.17, 0.20$ and $0 \le t \le 30$ on the same axes in part c.
Label each graph with its k value.
Discuss your findings.

Scenario 2
Specifications:
The population is finite. Let M be the size of the population.
The spread of virus is NOT controlled, i.e. no lockdown, no quarantine.
The rate of growth of the number of people infected with the virus is jointly proportional to the number of infected people N and the number of uninfected people $(M - N)$.

a. Scenario 2 can be modelled by the differential equation $\frac{dN}{dt} = \alpha N(M - N)$ where α is a constant.
Use the method of separation of variables and integration by partial fractions to show that
$\frac{N}{M - N} = \frac{N_0}{M - N_0} e^{\alpha M t}$ where N_0 is the number of infected people initially (i.e. when $t = 0$).

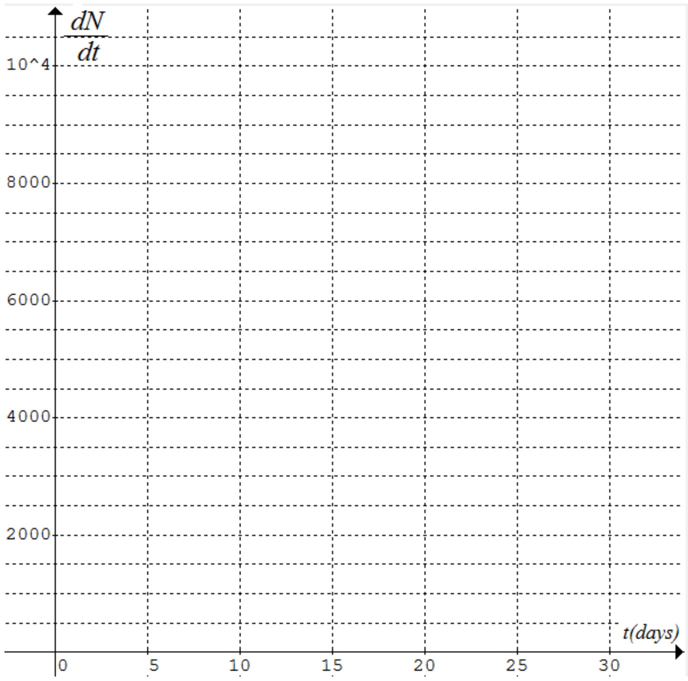
b. By transposition of $\frac{N}{M - N} = \frac{N_0}{M - N_0} e^{\alpha M t}$ show that $N = \frac{N_0 M}{N_0 + (M - N_0)e^{-\alpha M t}}$, and $N \to M$ as $t \to \infty$.

c. Let $N_0 = 10$, $M = 10000$ and $\alpha = 0.00004$. Sketch the graph of $N = \dfrac{N_0 M}{N_0 + (M - N_0)e^{-\alpha M t}}$, $t \geq 0$.



d. Let $N_0 = 10$, $M = 10000$ and $\alpha = 0.00004$.

Sketch the graph of the derivative of $N = \dfrac{N_0 M}{N_0 + (M - N_0)e^{-\alpha M t}}$, $t \geq 0$



e. Discuss the effects of the population being finite on (1) growth and (2) rate of growth of number of infected people.

Part II (60 – 85 minutes)

The rate of growth of the number of people infected with the virus is jointly proportional to the number of infected people N and the number of uninfected people $(M-N)$, i.e. $\frac{dN}{dt}=\alpha N(M-N)$.

The solution to the differential equation is $N(t)=\frac{N_0 M}{N_0+(M-N_0)e^{-\alpha M t}}$, $t\geq 0$.

Scenario 3
Specifications:
The spread of virus is controlled by STRICT ENFORCEMENT of household isolation.

Assumptions:
1. Each household has n members.
2. Initially (when strict household isolation is first enforced) there are N_0 infected people in a population.
3. Each of the N_0 infected people belongs to a different household.

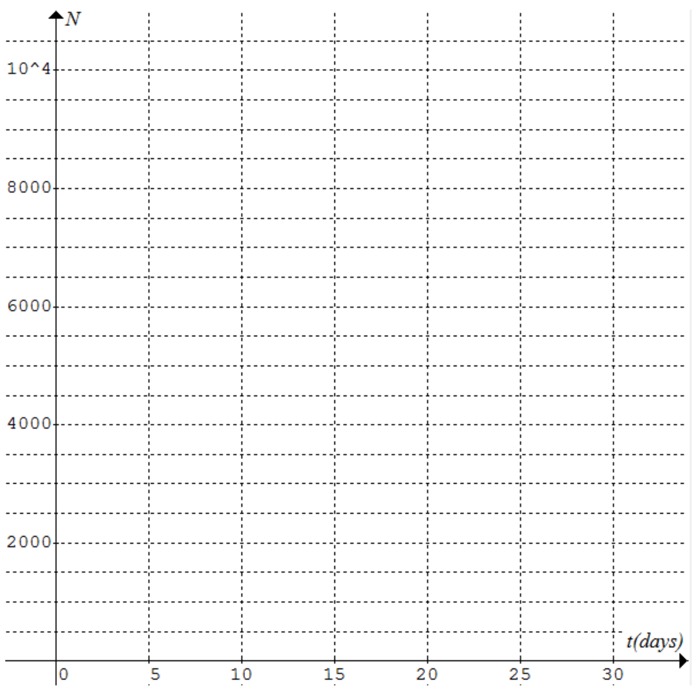
Under strict enforcement of household isolation and based on the above assumptions, the potential maximum number of infected people is $n\times N_0$, i.e. $M=nN_0$.

a. Write a differential equation, in terms of n and N_0, to model the spread of virus based on the above assumptions under strict enforcement of household isolation. Use α as the constant of proportionality.

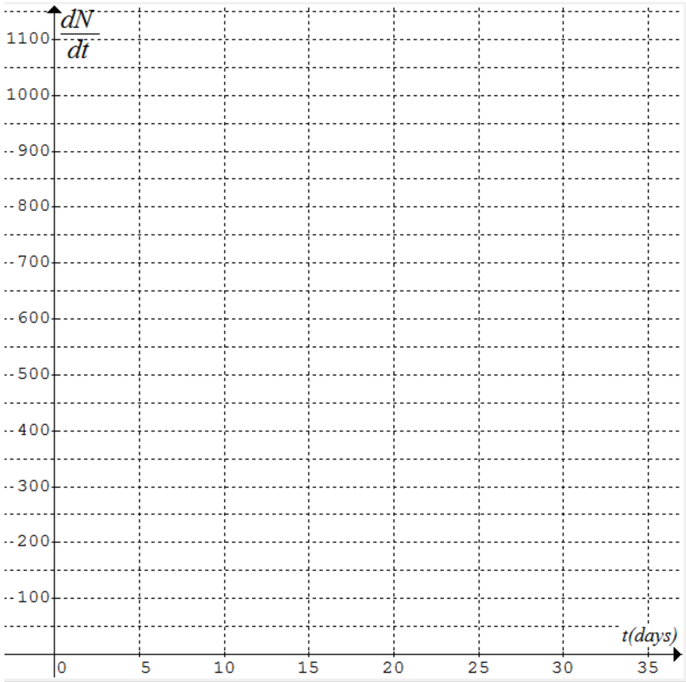
b. Write down the solution to the differential equation in terms of the constants α, n and N_0 only.

The value of N_0 depends on when strict household isolation is first enforced.
Early enforcement of strict household isolation results in low value of N_0.

c. Let $n=5$ and $\alpha=0.00004$.
Hint: Sketch the graphs of $N(t)$ for $N_0=200, 500, 1000, 2000$ on the same axes.



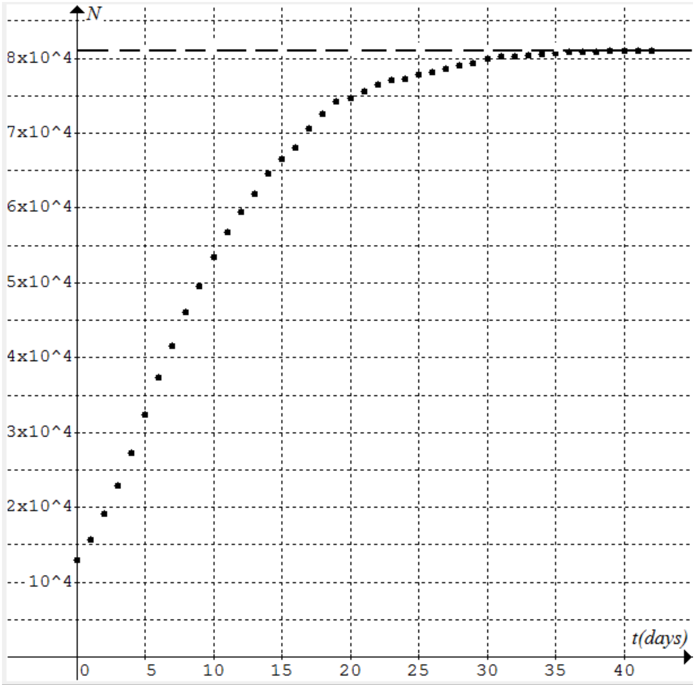
d. Sketch the graph of the derivative of $N(t)$ in part c for each of $N_0 = 200, 500, 1000, 2000$ on the same axes.



e. With reference to the graphs in parts c and d, discuss the effects of **timing** of strict household isolation enforcement on (1) growth and (2) rate of growth of number of infected people.

f. With reference to the graphs in parts c and d, discuss the effects of **timing** of strict household isolation enforcement on strains placed on hospitals in caring for patients infected with the virus.

The following graph is a plot of the number of infected people N in a region under strict household isolation enforcement.
The set of data was collected from the Daily Situation Reports published by World Health Organisation.



At $t = 0$, $N = N_0 = 12930$; as $t \to \infty$, $N \to 81050$.

Assume $N(t) = \dfrac{N_0 M}{N_0 + (M - N_0)e^{-\alpha M t}}$, and each of the N_0 infected people belongs to a different household.

g. Show that $N(t) \approx \dfrac{1.048 \times 10^9}{12930 + 68120 \times e^{-81050\alpha t}}$.

h. At $t = 10$, $N = 53510$.

Use this information to show that $\alpha \approx 0.0000028697$ and $N(t) \approx \dfrac{1.048 \times 10^9}{12930 + 68120 \times e^{-0.2326t}}$.

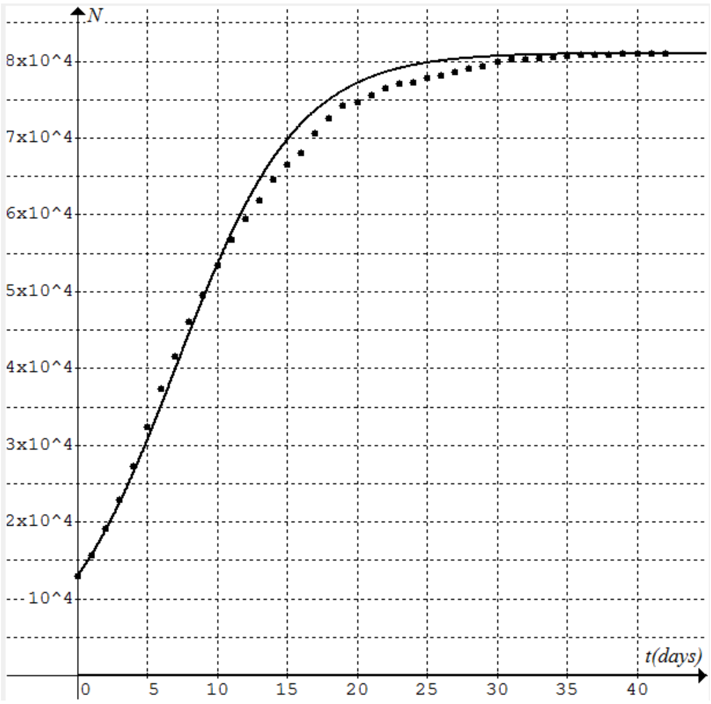
i. At $t = 15$, $N = 66570$.

Use this information to show that $\alpha \approx 0.0000026215$ and $N(t) \approx \dfrac{1.048 \times 10^9}{12930 + 68120 \times e^{-0.2125t}}$.

j. The graph of $N(t) \approx \dfrac{1.048\times10^{9}}{12930+68120\times e^{-0.2326t}}$ is shown below.

Sketch the graph of $N(t) \approx \dfrac{1.048\times10^{9}}{12930+68120\times e^{-0.2125t}}$ on the same axes.

Which one gives the closest fit to the given data points? Explain.



k. Determine the average number of people per household in the region.

l. Comment on changes to the above curve that could occur if household isolation was not strictly adhered to.

End of Task