2020 VCAA Specialist Mathematics Exam 2 Solutions

Use CAS whenever practical

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
D	B	A	E	A	C	D	A	B	D

11	12	13	14	15	16	17	18	19	20
C	C	E	B	E	D	B	E	B	A

Q1 Let $f'(x)=\frac{(x-2)((x+3)+(x-a))-(x-a)(x+3)}{(x-2)^2}=0$ and

$x=0$, $a=\frac{6}{5}$ **D**

Q2 Range of $b\cos^{-1}(x)-a$ is $[-a, b\pi-a]$

Given $a<\frac{b\pi}{2}$, range of $\left|b\cos^{-1}(x)-a\right|$ is $[0, b\pi-a]$ **B**

Q3 For $230<t\le 260$, $\frac{v-0}{t-260}=\frac{10-0}{230-260}$, $v=\frac{1}{3}(260-t)$ **A**

Q4 $f(g(x))=\frac{\sqrt{\operatorname{cosec}^2 x-1}}{\operatorname{cosec}^2 x}=\frac{\sqrt{\cot^2 x}}{\operatorname{cosec}^2 x}=\frac{\cos x}{\sin x}\times\sin^2 x$

$=\frac{1}{2}\sin 2x$, $f\left(g\left(\frac{\pi}{4}\right)\right)=\frac{1}{2}$, range $\left(0,\frac{1}{2}\right]$ **E**

Q5 $\frac{4(a^2+b^2)}{4a^2}=1+\left(\frac{b}{a}\right)^2=1+\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)^2$ **A**

Q6 $P(z)=z^3+az^2+bz+c=(z+2)(z-3i)(z+3i)$

.: $c=2(-3i)(3i)=18$ **C**

Q7 $\frac{1}{ax(x^2+b)}=\frac{1}{ax\left(x^2-\left(\sqrt{|b|}\right)^2\right)}=\frac{A}{x}+\frac{B}{x+\sqrt{|b|}}+\frac{C}{x-\sqrt{|x|}}$ **D**

Q8 $(y-ix)^{14}=(-i(x+iy))^{14}=i^{14}(x+iy)^{14}=-1(x+iy)^{14}$

$=-a-ib$ **A**

Q9 **B**

Q10 Volume is $50+(2-5)t=50-3t$ at time t. **D**

Q11 $u=\tan x$, $\frac{du}{dx}=\sec^2 x$, $\int_1^{\sqrt{3}}\frac{du}{u^2-3u+2}=\int_1^{\sqrt{3}}\left(\frac{1}{u-2}-\frac{1}{u-1}\right)du$ **C**

Q12 **C**

Q13 Let $\tilde{b}=m\tilde{a}+n\tilde{c}$.: $m+n=\lambda$, $2m=3$ and $-m+n=2$

.: $m=\frac{3}{2}, n=\frac{7}{2}, \lambda=5$ **E**

Q14 $\hat{d}=\frac{1}{7}\left(2\tilde{i}-3\tilde{j}-6\tilde{k}\right)$, $\tilde{F}.\tilde{d}=\frac{1}{7}(2-18+108)=\frac{92}{7}$ **B**

Q15 Resultant force $\tilde{R}=6\tilde{i}+3\tilde{j}$, $\tilde{a}=\frac{\tilde{R}}{m}=2\tilde{i}+\tilde{j}$.: $\tilde{v}=2t\,\tilde{i}+t\,\tilde{j}$

$\tilde{r}=(t^2+1)\tilde{i}+\left(\frac{t^2}{2}+1\right)\tilde{j}$.: $y=\frac{x+1}{2}$ **E**

Q16 $\tilde{a}.\tilde{b}=ab\cos\theta$, $\cos\theta=\frac{1}{9}$, $\sin\theta=\frac{4\sqrt{5}}{9}$

.: $\sin 2\theta=2\sin\theta\cos\theta=\frac{8\sqrt{5}}{81}$ **D**

Q17 $v=\frac{1}{x}$, $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{1}{2}\frac{d}{dx}\left(\frac{1}{x^2}\right)=-\frac{1}{x^3}$; $x=2$, $a=-\frac{1}{8}$ **B**

Q18 $T\cos\alpha=F$, $T\sin\alpha=mg$.: $\tan\alpha=\frac{mg}{F}$ **E**

Q19 $\Delta\tilde{p}=m\Delta\tilde{v}=0.02\left(\left(2\tilde{i}-7\tilde{j}\right)-\left(2\tilde{i}-10\tilde{j}\right)\right)=0.06\tilde{j}$ **B**

Q20 $\tilde{a}=\frac{\tilde{R}}{m}=\frac{{}^{+}30+{}^{-}2g}{2}={}^{+}5.2$ **A**

SECTION B

Q1a $t=\frac{\pi}{6}$, $x=\sqrt{3}$, $y=\frac{3\sqrt{3}}{2}$, distance $=\sqrt{x^2+y^2}=\frac{\sqrt{39}}{2}$

Q1bi $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-3\sin t}{4\cos 2t}$; $t=\pi$, $x=0$, $y=-3$ and $\frac{dy}{dx}=0$

.: tangent to the curve at $(0,-3)$ is the horizontal line $y=-3$.

Q1bii $\tilde{r}=2\sin 2t\,\tilde{i}+3\cos t\,\tilde{j}$, $\tilde{v}=4\cos 2t\,\tilde{i}-3\sin t\,\tilde{j}$

$t=\pi$, $\tilde{v}=4\tilde{i}$

Q1biii $t=\pi$, $\tilde{a}=-8\sin 2t\,\tilde{i}-3\cos t\,\tilde{j}=3\tilde{j}$, $|\tilde{a}|=3$

Q1c $2\sin 2t=0$ and $3\cos t=0$, $t=\frac{\pi}{2}$

Q1d $d=\int_0^{\frac{\pi}{6}}\sqrt{(4\cos 2t)^2+(-3\sin t)^2}\,dt\approx 1.804$

Q2a The line is a perpendicular bisector of the line joining u and v.

Gradient of the line joining u and v is $\frac{-1-(-3)}{-2-(-4)}=1$

Midpoint $\left(\frac{-2+(-4)}{2},\frac{-1+(-3)}{2}\right)=(-3,-2)$

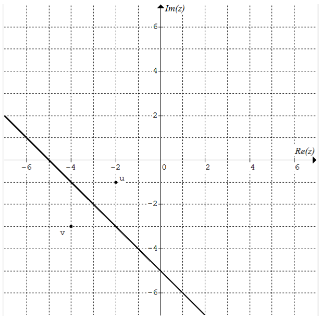
Equation: $\frac{y-(-2)}{x-(-3)}=-1$, $y=-x-5$

Q2a Alternatively, let $z = x + iy$,

$\left|(x+2)+i(y+1)\right|^2 = \left|(x+4)+i(y+3)\right|^2$

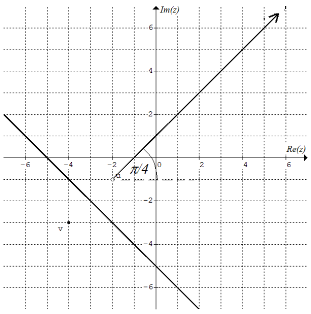
$(x+2)^2 + (y+1)^2 = (x+4)^2 + (y+3)^2$, $y = -x-5$

Q2b



Q2c The line is a perpendicular bisector of the line joining u and v.

Q2di



Q2dii $f : (-2, \infty) \to R$, $f(x) = x + 1$

Q2e $r = \left|z_c - (-5i)\right| = \left|z_c - u\right| = \left|z_c - v\right|$

.: $a^2 + (b+5)^2 = (a+2)^2 + (b+1)^2$ and

$a^2 + (b+5)^2 = (a+4)^2 + (b+3)^2$.: $a = -\frac{5}{3}$ and $b = -\frac{10}{3}$

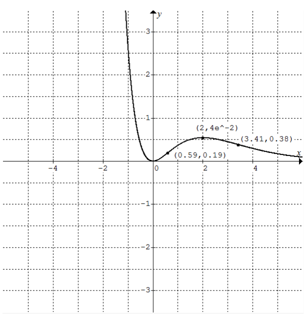
.: $z_c = -\frac{5}{3} - \frac{10}{3}i$ and $r = \sqrt{a^2 + (b+5)^2} = \frac{5\sqrt{2}}{3}$

Q3a $f(x) = x^2 e^{-x}$. $f'(x) = -x^2 e^{-x} + 2xe^{-x}$ Let $f'(x) = 0$

.: $x = 0$ and $y = 0$; $x = 2$ and $y = 4e^{-2}$.: $(0, 0)$, $(2, 4e^{-2})$

Q3b As $x \to \infty$, $y \to 0^+$, asymptote: $y = 0$

Q3c At the inflection points, $f''(x) = 0$, $(0.59, 0.19)$, $(3.41, 0.38)$



Q3d $g''(x) = x^{n-2} e^{-x}\left((x-n)^2 - n\right)$

Q3ei Non-zero values of x, $(x-n)^2 = n$, $x = n \pm \sqrt{n}$ for $n \geq 2$

Q3eii

Number of inflection pts	Value of $n \geq 0$
0	0
1	1
2	n (even integer)
3	n (odd integer)

Q4a $\tilde{\dot{r}}_A = \left(-25\pi\cos\frac{\pi t}{6}\right)\tilde{i} + \left(\frac{100\pi}{3}\sin\frac{\pi t}{6}\right)\tilde{j}$

Speed $= \left|\tilde{\dot{r}}_A\right| = \frac{100\pi}{3}\sqrt{\frac{9}{16}\cos^2\frac{\pi t}{6} + \sin^2\frac{\pi t}{6}} = \frac{100\pi}{3}\sqrt{1 - \frac{7}{9}\cos^2\frac{\pi t}{6}}$

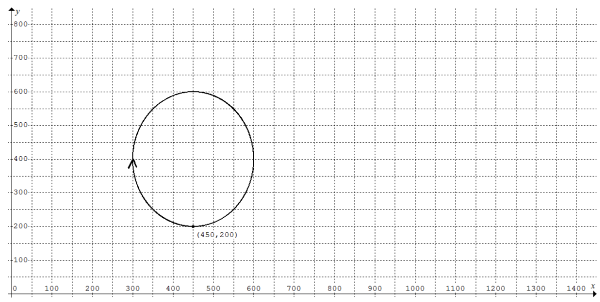
Maximum speed $= \frac{100\pi}{3}\sqrt{1} = \frac{100\pi}{3}$ when $\cos\frac{\pi t}{6} = 0$

Q4bi $\left(\sin\frac{\pi t}{6}\right)^2 = \left(\frac{450-x}{150}\right)^2 = \left(\frac{x-450}{150}\right)^2 = \frac{(x-450)^2}{22500}$ and

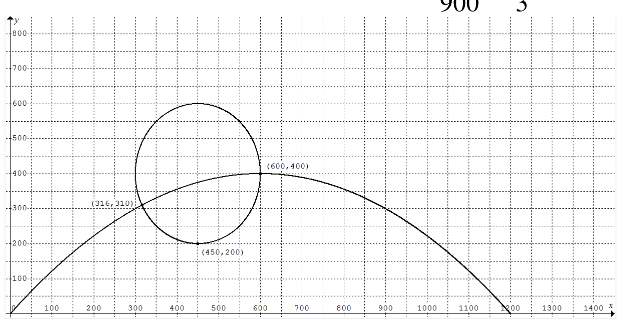
$\left(\cos\frac{\pi t}{6}\right)^2 = \left(\frac{400-y}{200}\right)^2 = \left(\frac{y-400}{200}\right)^2 = \frac{(y-400)^2}{40000}$

.: $\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = 1$

Q4bii



Q4c $x = 30t$, $y = -t^2 + 40t$, $0 \leq t \leq 40$, $y = -\frac{x^2}{900} + \frac{4x}{3}$

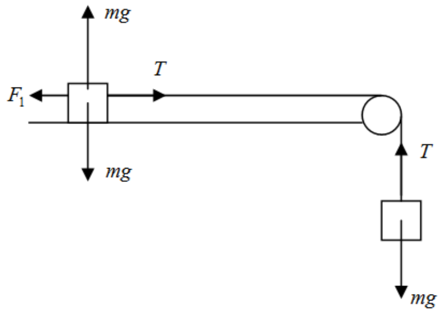


Q4e $30t = 450 - 150\sin\frac{\pi t}{6}$, same x-coord. when $t \approx 12.85$

$-t^2 + 40t = 400 - 200\cos\frac{\pi t}{6}$, same y-coord. when $t = 10$ or ≈ 14.73

At no time do they have the same x and y-coordinates .: no contact

Q5a



Q5bi $mg - F_1 = (m+m)a$, $mg - kmg = 2ma$, $a = \frac{g(1-k)}{2}$

Q5bii $a > 0$ and $k > 0$, $\frac{g(1-k)}{2} > 0$.: $0 < k < 1$

Q5c $u = 0$, $a = \frac{g(1-k)}{2}$, $s = 5$, $s = ut + \frac{1}{2}at^2$, $t = 2\sqrt{\frac{5}{g(1-k)}}$

Q5d $v^2 = u^2 + 2as$, $v_B = \sqrt{5g(1-k)}$

Q5e $a = -\frac{F_2}{m} = -(0.075g + 0.4v^2) = -0.4(0.1875g + v^2)$

.: $\frac{1}{2}\frac{dv^2}{dx} = -0.4(0.1875g + v^2)$, $-\frac{1}{0.8}\frac{dv^2}{dx} = 0.1875g + v^2$

$-0.8\frac{dx}{dv^2} = \frac{1}{0.1875g + v^2}$

$-0.8x = \int \frac{1}{0.1875g + v^2} dv^2 = \log_e(0.1875g + v^2) + c$

Let $x = 0$ at B and $v = v_B = 2.5$. $c = -\log_e(0.1875g + 2.5^2)$

.: $-0.8x = \log_e\left(\frac{0.1875g + v^2}{0.1875g + 2.5^2}\right)$

When $v = 0$, $x \approx 1.85$.: distance from C $\approx 5 - 1.85 = 3.15$.

Please inform mathline@itute.com re conceptual and/or mathematical errors

