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2021 Mathematical Methods

Year 12 Application Task

Time allowed: 4 hours plus

Theme: Designing a bicycle path through a park

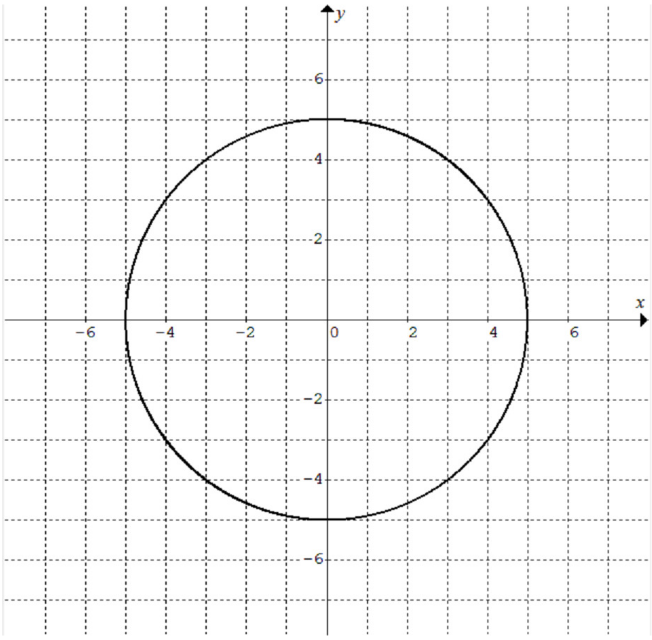
Assumed knowledge: Functions, e.g. power functions, polynomial functions, trigonometric functions; transformations; calculus; solving equations; graphs; CAS.

Several functions and their features/behaviours are analysed in Part I and Part II.
In Part III make use of the results in Part I and Part II to design a bicycle path through a park.

**** To ensure accuracy of your solutions/answers use appropriate number of decimal places to express numerical values.*

Part I (80 – 90 minutes)

The equation of the circle shown below is $x^2 + y^2 = 25$.



a. Find the equation of a quadratic function which passes through $(-4, 2)$ and has its vertex at $(0,5)$.

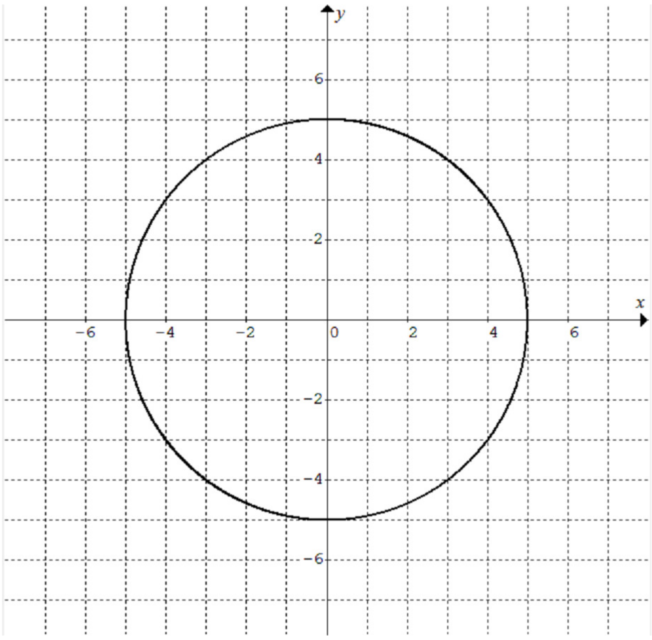
Express the equation in the form $y = -\left(\frac{x}{a}\right)^2 + c$ where $a > 0$.

Find the coordinates of the intersection between the quadratic function and the circle.
Sketch the graph of the quadratic function on the same Cartesian plane as the circle shown above.

b. Find the equation of a cosine function which passes through $(-4, 2)$ and has its vertex at $(0,5)$.

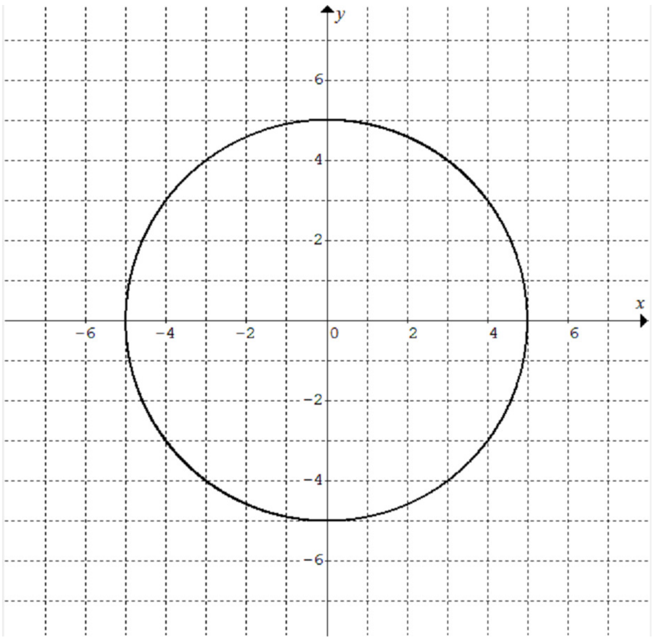
Express the equation in the form $y = A\cos\left(\frac{x}{a}\right)$ where $a > 0$.

Find the coordinates of the intersection between the cosine function and the circle.
Sketch the graph of the cosine function on the same Cartesian plane as the circle shown below.



c. A function has the equation $y = 10\sqrt{1-\left(\frac{x}{a}\right)^2} - c$ where $a > 0$.

Find the equation of the function if it passes through $(-4, 2)$ and has its vertex at $(0,5)$.
Find the coordinates of the intersections between the function and the circle.
Sketch the graph of the function on the same Cartesian plane as the circle shown below.



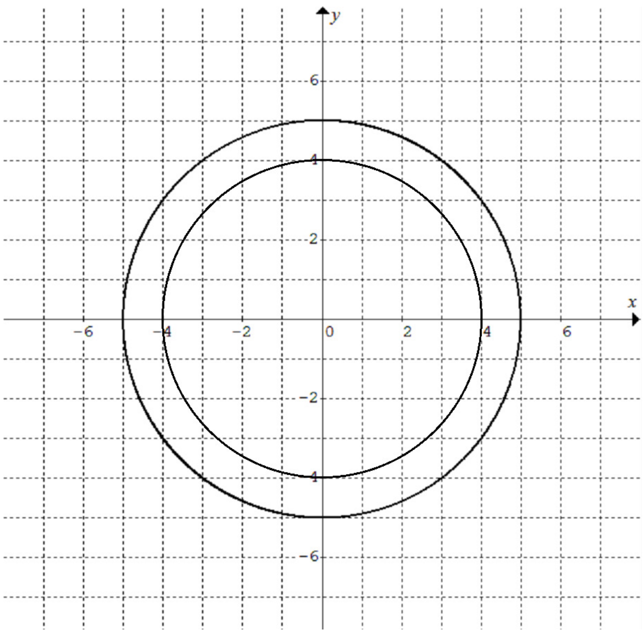
In parts d, e and f, you explore the functions with equations $y=-\left(\frac{x}{a}\right)^2+c$, $y=A\cos\left(\frac{x}{a}\right)$ *and* $y=10\sqrt{1-\left(\frac{x}{a}\right)^2}-c$ *respectively for* $x\le 0$.

The graphs of circles $x^2+y^2=16$ *and* $x^2+y^2=25$ *accompany each part.*

d (i) Investigate the effects of changing parameter a on the function with equation $y=-\left(\frac{x}{a}\right)^2+c$, $x\le 0$ in relation to its shape, vertex and number of intersections with circle $x^2+y^2=25$, and comment.

Hint: Let $c=5$. Use several values of a, e.g. $a=8$, 5, 2 and compare their graphs for $y=-\left(\frac{x}{a}\right)^2+c$.

Label each graph with its value of a.

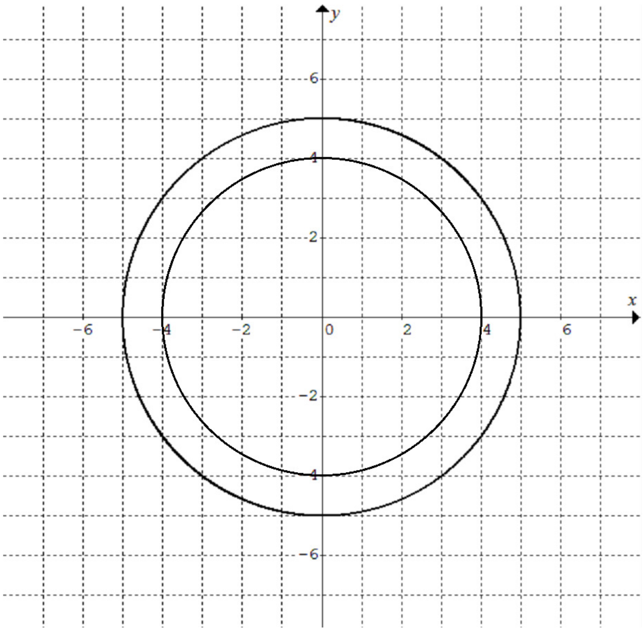


d (ii) Find the values of a such that there is more than one intersection with circle $x^2+y^2=25$ **but** no more than one intersection with circle $x^2+y^2=16$.

e (i) Investigate the effects of changing parameter a on the function with equation $y = A\cos\left(\frac{x}{a}\right)$, $x \le 0$ in relation to its shape, vertex and number of intersections with circle $x^2 + y^2 = 25$, and comment.

Hint: Let $A = 5$. Choose three values of $a \in [2, 8]$ and compare their graphs for $y = A\cos\left(\frac{x}{a}\right)$.

Label each graph with its value of a.

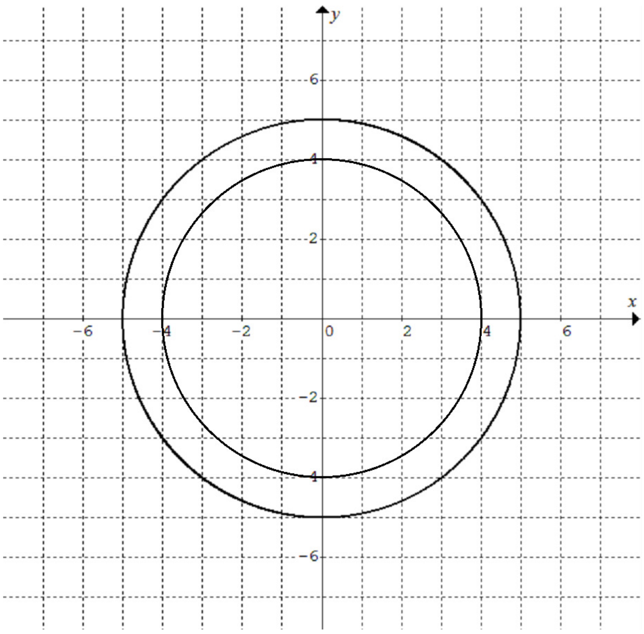


e (ii) Find the values of a such that there is more than one intersection with circle $x^2 + y^2 = 25$ **but** no more than one intersection with circle $x^2 + y^2 = 16$.

f (i) Investigate the effects of changing parameter a on the function with equation $y = 10\sqrt{1-\left(\frac{x}{a}\right)^2} - c$, $x \le 0$ in relation to its shape, vertex and number of intersections with circle $x^2 + y^2 = 25$, and comment.

Hint: Let $c = 5$. Choose three values of $a \in [2, 8]$ and compare their graphs for $y = 10\sqrt{1-\left(\frac{x}{a}\right)^2} - c$.

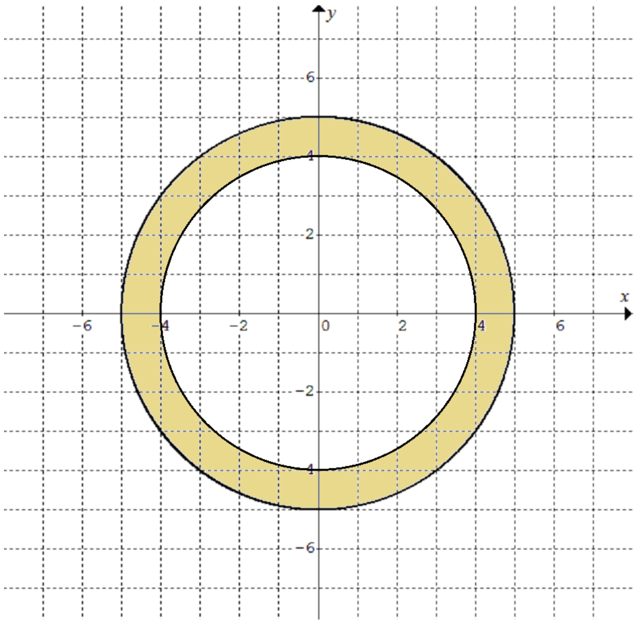
Label each graph with its value of a.



f (ii) Find the values of a such that there is more than one intersection with circle $x^2 + y^2 = 25$ **but** no more than one intersection with circle $x^2 + y^2 = 16$.

Part II (80 – 90 minutes)

Two circles of radii 4 and 5 are shown below. The region between them is shaded.



a. A cubic polynomial has equation $y = ax^3 + bx^2 + cx + d$. It has a turning point at $\left(0, \dfrac{48}{10}\right)$.

Show that $c = 0$ and $d = \dfrac{48}{10}$.

b. The cubic polynomial passes through points $(2, 4)$ and $\left(\dfrac{9}{2}, 0\right)$. Show that $a = -\dfrac{2}{135}$ and $b = -\dfrac{23}{135}$.

c. Sketch on the diagram above the graph of the cubic polynomial for $x \geq 0$.

d. The graph of the cubic polynomial for $x \geq 0$ is reflected in both axes.
State the equation, domain and range of the image after the transformations.

e. The graph of the *original* cubic polynomial for $x \geq 0$ is dilated from the *x*-axis such that the image of the original turning point $\left(0, \frac{48}{10}\right)$ is $\left(0, \frac{49}{10}\right)$. Determine the equation of the image after the dilation.

f. Find the image of each of the points $(2, 4)$ and $\left(\frac{9}{2}, 0\right)$ after the dilation.

g. The graph of the *original* cubic polynomial for $x \geq 0$ and the graph of $y = -\left(\frac{x}{a}\right)^2 + c$ for $x \leq 0$ are joined smoothly at $\left(0, \frac{48}{10}\right)$. The graph of $y = -\left(\frac{x}{a}\right)^2 + c$ passes through $(-5, 0)$.
Determine the value of a and c.

h. The graph of $y = A\cos\left(\frac{x}{a}\right)$ for $x \leq 0$ and the graph of $y = 10\sqrt{1 - \left(\frac{x}{k}\right)^2} - c$ for $x \geq 0$ are joined smoothly at $\left(0, \frac{48}{10}\right)$. The graph of $y = A\cos\left(\frac{x}{a}\right)$ passes through $(-5, 0)$ and the graph of $y = 10\sqrt{1 - \left(\frac{x}{k}\right)^2} - c$ passes through $(5, -1)$.
Determine the value of each of A, a, c and k.

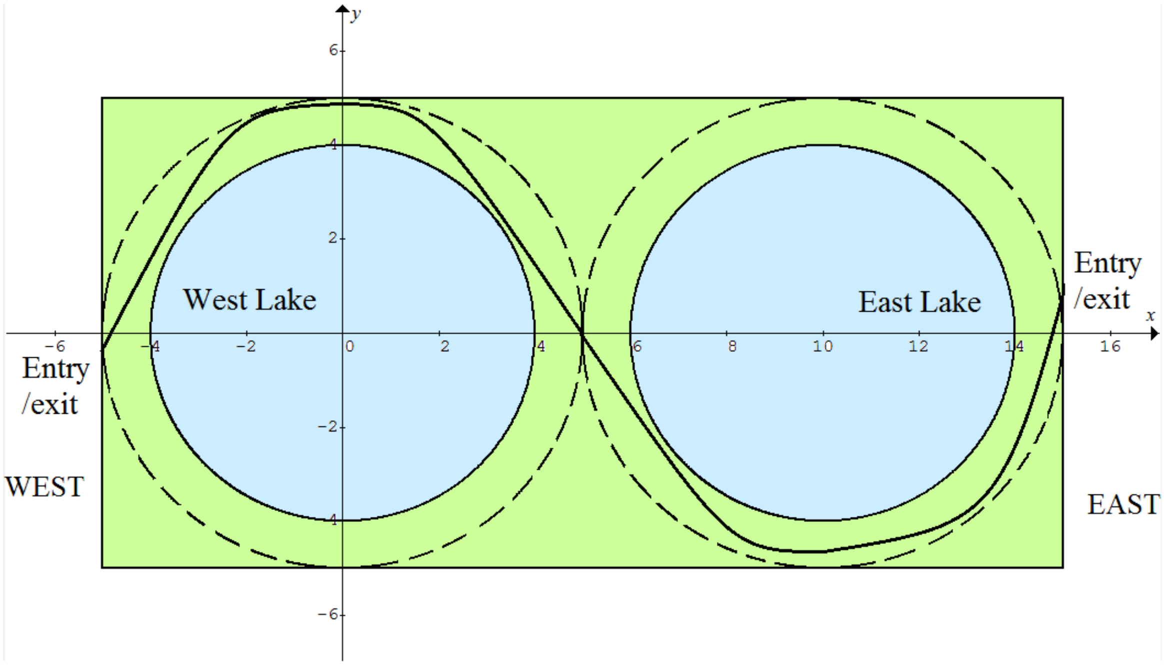
The length of a curve can be determined by the formula $\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ where x_1 and x_2 are the x-coordinates of the end points of the curve.

i. Calculate the total length of the curve in part g **in the shaded region** between the two circles.

j. Calculate the total length of the curve in part h **in the shaded region** between the two circles.

Part III (80 – 90 minutes)

The following diagram shows a rectangular park 20 km by 10 km. Two circular artificial lakes of the same radius 4 km are also shown. The dotted circles (radius 5 km) in the diagram are for your reference only. A plausible bicycle path inside the park is drawn on the diagram as an example.

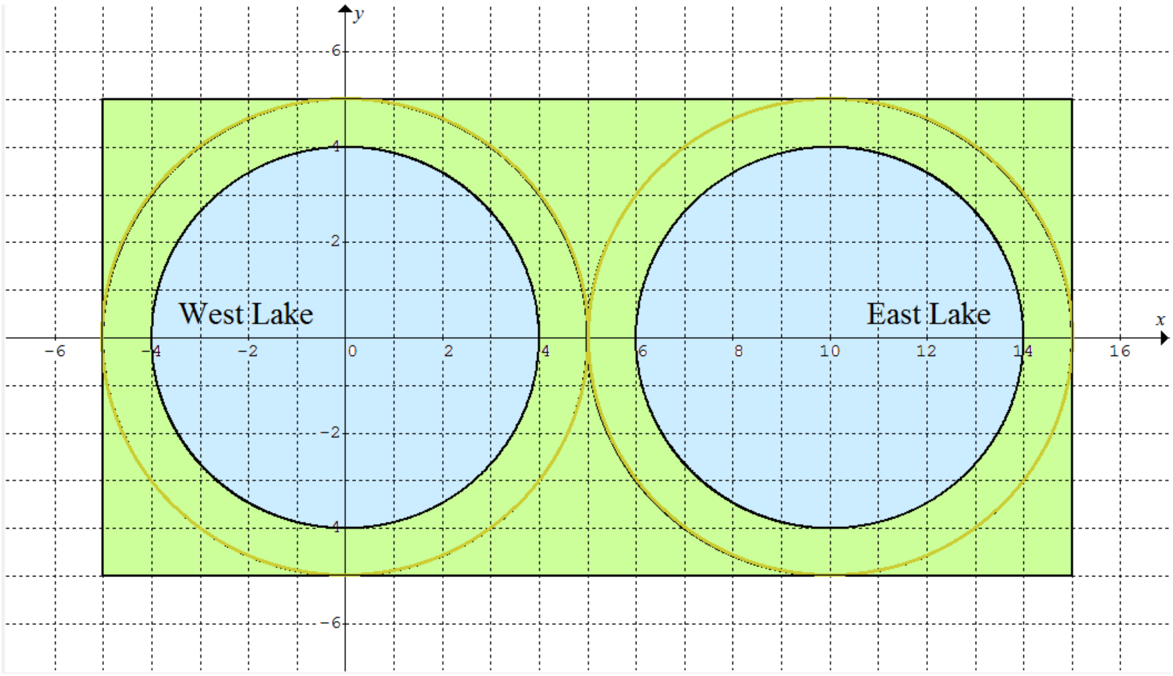


a. Your task is to design a bicycle path inside the park satisfying the following specifications.

1. There are two entry/exit points. One is on the west boundary of the park, the other on the east boundary.
2. An ideal path should be
   * as simple as possible
   * as long as possible
   * no more than 1 km from a lake except near the entry/exit points
   * at least 0.1 km from the lakes and the boundaries except near the entry/exit points.
3. Use $y = A\cos\left(\frac{x}{a}\right)$ for the middle section (from the north of West Lake to the south of East Lake) and **two** other types of functions explored in Part I and Part II to make up the path.
4. The different sections of the path should be joined smoothly.

To complete part a, you are required

1. to write a piecewise function showing the different sections
2. to state the value of $\frac{dy}{dx}$ at each joining point
3. to state the coordinates of each entry/exit point
4. to state the length of each section inside the park and the total length
5. to sketch the path on the following diagram



y
x
West Lake
East Lake
-6
-4
-2
0
2
4
6
8
10
12
14
16
6
4
2
-2
-4
-6

b. **Replace** $y = A\cos\left(\frac{x}{a}\right)$ in part a with $y = -\left(\frac{x}{a}\right)^2 + c$ and its transformation joined smoothly at $(5, 0)$ for the middle section (from the north of West Lake to the south of East Lake).
The other sections of the path should be joined smoothly to the middle section.
Note: The only difference between the paths in part a and part b is the middle section.

To complete part b you are required

1. to write a piecewise function showing the different sections
2. to state the total length
3. to verify the sections are joined smoothly
4. to verify algebraically the middle section is within 1 km from a lake except at $(5, 0)$.

End of Application Task