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2021 Mathematical Methods

Year 12

Modelling Task

Time allowed: 2 hours plus

Modelling Task

Theme: Irrigation channels

Assumed knowledge:

Functions and graphs, transformations, equations, differentiation and integration, area of region bounded by curves, CAS

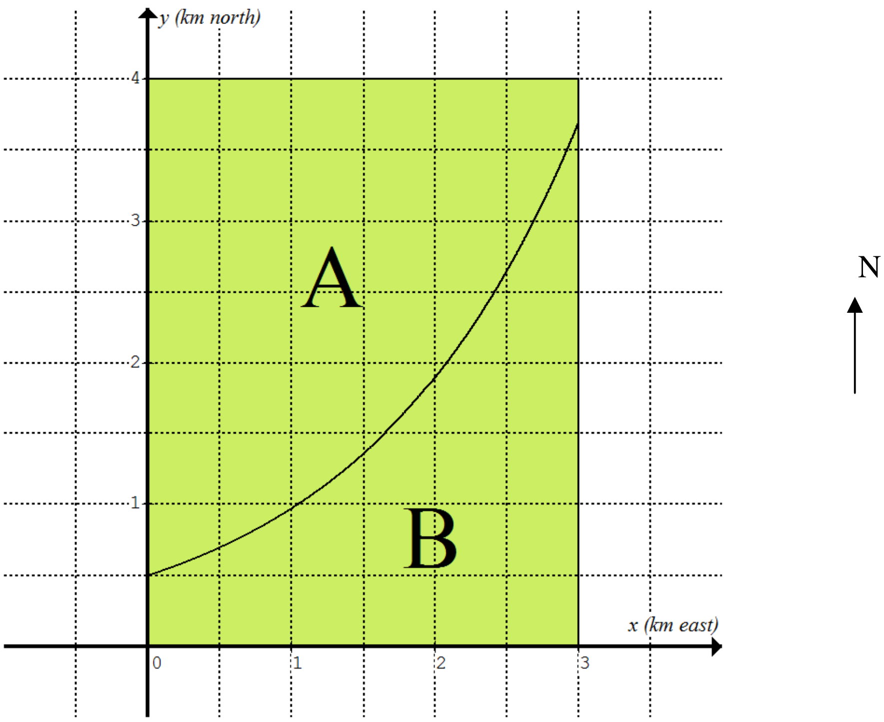
Specifications:

An irrigation channel runs from west to east across a flat rectangular property as shown in the diagram below.

The channel divides the property into Region A (north of the channel) and Region B (south of the channel).

The channel is modelled by the equation $y = \frac{1}{2}e^{kx}$ where $k \in R$.

Distance/length is measured in kilometres (km), area in kilometre squares (km^2).



Part I (75 minutes)

Let $k = \frac{2}{3}$ for part a and part b.

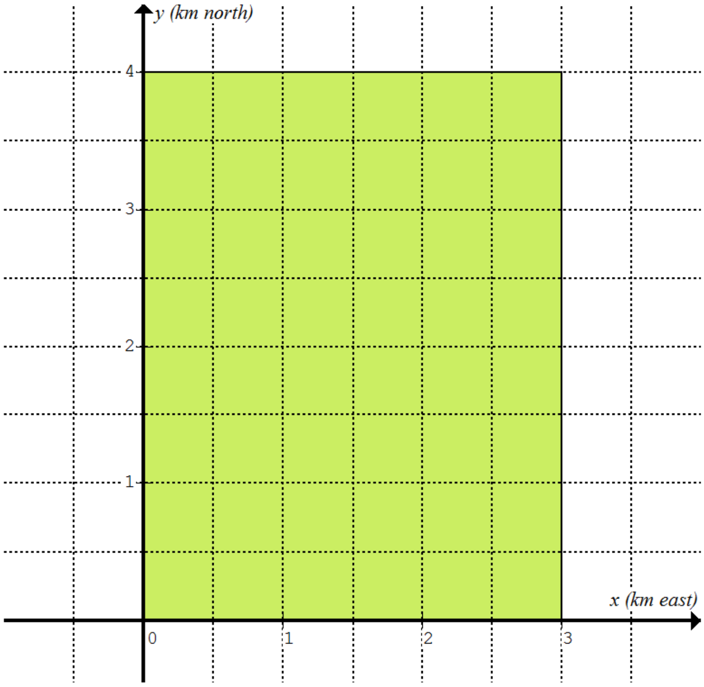
a. Show that the area of Region A is $\frac{3}{4}\left(17 - e^2\right)$.

b. The area of Region A is α times the area of Region B. Show that $\alpha = \frac{16}{e^2 - 1} - 1$.

c. Investigate the effects of changing the value of k on the irrigation channel and the areas of the two regions.

Suggestion: Sketch the graph of the channel for $k = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$.

Label each graph with its k value.



Comment:

d. Explain whether or not the channel modelled by the equation $y = \frac{1}{2}e^{kx}$ crosses the southern boundary of the rectangular property.

The channel is at least $\frac{1}{2}$ km from the northern boundary and more than $\frac{1}{4}$ km from the southern boundary. It cuts across the eastern boundary.

e. Determine the possible values of k in exact form.

f. Show that $e^{3k} = 12k + 1$ if the area of Region A **equals** the area of Region B.
Find the value of k, correct to 3 decimal places, and compare with the possible values of k in part e.

g. The channel described in part a, $y = \frac{1}{2}e^{\frac{2}{3}x}$, is shifted to the south by $\frac{1}{4}$ km.

Now the area of Region A is β times the area of Region B, where $\beta = \frac{16}{e^2 - p} - q$ and $p, q \in R$. Determine the values of p and q.

h. If the channel described in part a, $y=\frac{1}{2}e^{\frac{2}{3}x}$, is shifted to the east by c km, determine the possible values of c in exact form.

i. If the channel described in part a, $y=\frac{1}{2}e^{\frac{2}{3}x}$, is shifted to the east by c km such that the area of Region A is **two** times the area of Region B, determine the value of c (correct to 3 decimal places).

End of Part I

Part II (60 minutes)

Information from Part I:

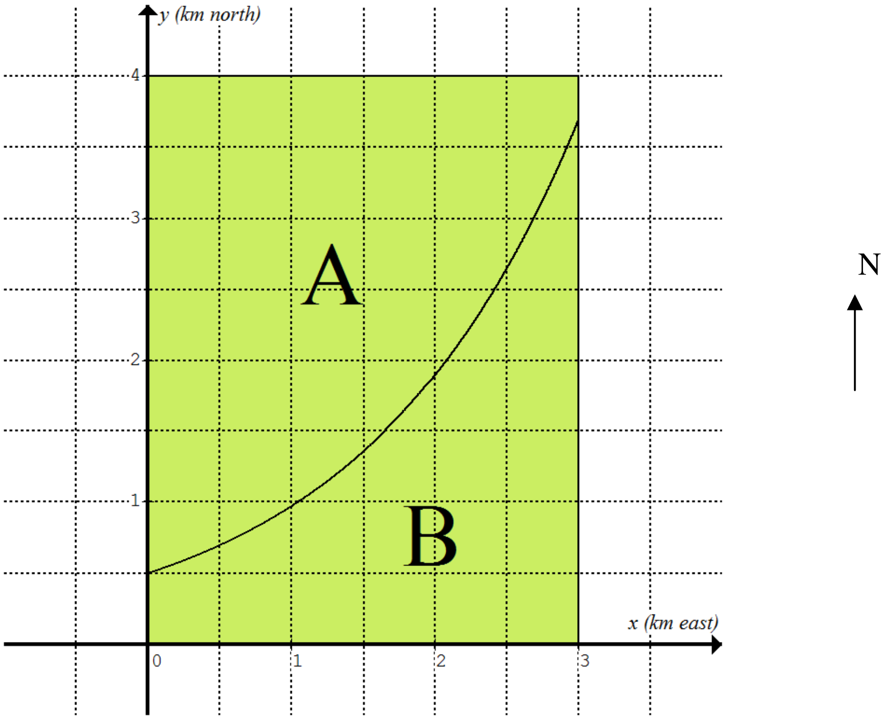
An irrigation channel runs from west to east across a flat rectangular property as shown in the diagram below.

The channel divides the property into Region A (north of the channel) and Region B (south of the channel).

The channel is modelled by the equation $y = \frac{1}{2}e^{\frac{2}{3}x}$.

Consider this channel as the main channel. Label it as Channel M.

Distance/length is measured in kilometres (km), area in km^2.



Area of Region A is $\frac{3}{4}\left(17 - e^2\right)$.

Area of Region B is $\frac{3}{4}\left(e^2 - 1\right)$.

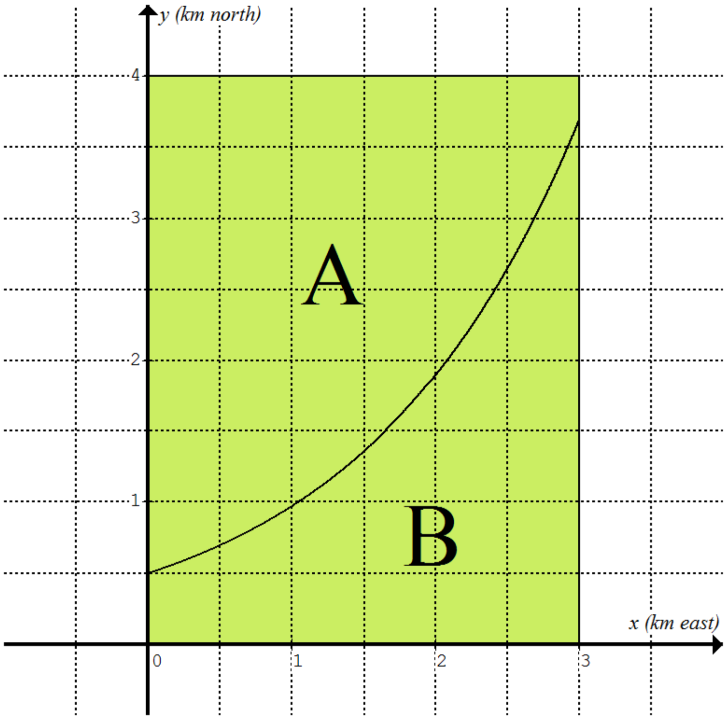
For Part II, correct answers to 3 decimal places unless stated otherwise.

Another channel in Region A (labelled as Channel A) branches off from Channel M at $x \approx 0.493$. The equation of Channel A is $y = (x - 0.100)^3 + h$ where $h \in R$.

a. Show that $h \approx 0.634$.

b. Show that $\frac{dy}{dx}$ for Channel M and Channel A are equal at $x \approx 0.493$.

c. Sketch the graph of Channel A in Region A only. Coordinates of endpoints are required.



d. Determine the area of the region inside Region A bounded by Channel A and Channel M.

e. Determine the area of the remaining region inside Region A, which is outside the region described in part d.

Another channel in Region B (labelled as Channel B) branches off from Channel M at (v, w) and extends eastward to the boundary.

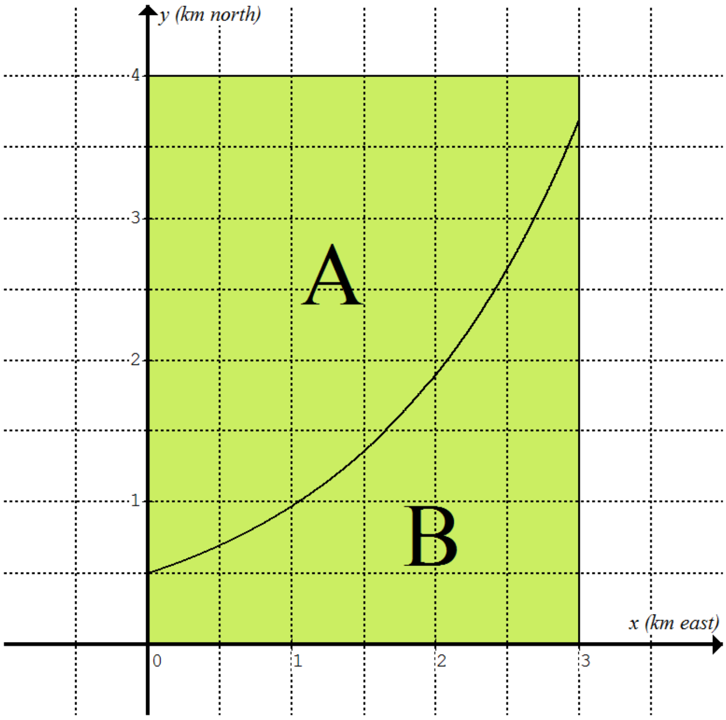
At (v, w), $\frac{dy}{dx}$ for Channel M and Channel B are **equal**.

The equation of Channel B is $y = 1.2\cos(nx - r)$ where $n \in R$ and $1.380 < r < 1.390$.

Choose a value for r.

f. Find the values of n, v and w for your chosen r value graphically, given a rough estimate of n is 0.7.
(Hint: Use $n = 0.7$ and your chosen r value to sketch the graph of $y = 1.2\cos(nx - r)$.
Adjust the value of n such that $y = 1.2\cos(nx - r)$ and $y = \frac{1}{2}e^{\frac{2}{3}x}$ intersect at only one point inside the rectangular property.)

g. Sketch the graph of Channel B in Region B only for your chosen r value.
Coordinates of endpoints are required.



h. Using the values of n, v and w for your chosen r found in part f, determine the area of the region inside Region B bounded by the southern boundary, Channel M and Channel B.

End of Part II