2021 NSW ESA Mathematics Advanced Solutions

Section I

1	2	3	4	5	6	7	8	9	10
B	C	D	C	A	D	A	C	B	B

Q2 $1\times 0.6+2\times 0.3+3\times 0.1=1.5$ C

Q3 $f(x)$ is the transformation of $\ln(x)$ under a reflection in the y-axis followed by a translation of 1 unit to the right. D

Q4 Check the cumulative downloads up to (including) Day 10. C

Q6 $\frac{3}{8}\times\frac{5}{7}+\frac{5}{8}\times\frac{3}{7}=\frac{15}{28}$ D

Q7 $f(0)<0$, $f'(3)=0$, $f''(-2)>0$.: $f(0)<f'(3)<f''(-2)$ A

Q9 $g(x)$ is even, $g(-x)=g(x)$, $g'(-x)=-g'(x)$
For odd f, $f(g(x))$ is even, tangents at $x=-k, k$ intersect at the y-axis.
$h'(x)=f'(g(x))g'(x)$
At $x=k$, $h'(k)=f'(g(k))g'(k)=m$
At $x=-k$, $h'(-k)=f'(g(-k))g'(-k)=-f'(g(k))g'(k)=-m$ B

Q10 $0<ma<1$.: $0<m<\frac{1}{a}$; $0<a<2\pi$.: $\frac{1}{2\pi}<\frac{1}{a}$
Line joining O and $(2\pi, 1)$ is below the tangent line $y=mx$
.: $\frac{1}{2\pi}<m$ B

Section II

Q11 $x+\frac{x-1}{2}=9$, $2x+x-1=18$, $3x=19$, $x=\frac{19}{3}$

Q12a $XY=16\cos 30°\approx 13.86\,\text{cm}$

Q12b Area $=\frac{1}{2}\pi\times 8^2-\frac{1}{2}\times 13.86\times 16\sin 30°\approx 45.1\,\text{cm}^2$

Q13 $y=x\tan x$, $\frac{dy}{dx}=x\sec^2 x+\tan x=\frac{x}{\cos^2 x}+\tan x$
At $x=\frac{\pi}{3}$, gradient $=\frac{4\pi}{3}+\sqrt{3}$

Q14 $S_{43}=\frac{43}{2}(2\times 5+42d)=2021$, $d=2$

Q15 $\int_{-2}^{0}(2x+4)^{\frac{1}{2}}\,dx=\left[\frac{2(2x+4)^{\frac{3}{2}}}{3\times 2}\right]_{-2}^{0}=\frac{8}{3}$

Q16 $f(x)=x^2-2x^3$, $f'(x)=2x(1-3x)>0$, $0<x<\frac{1}{3}$

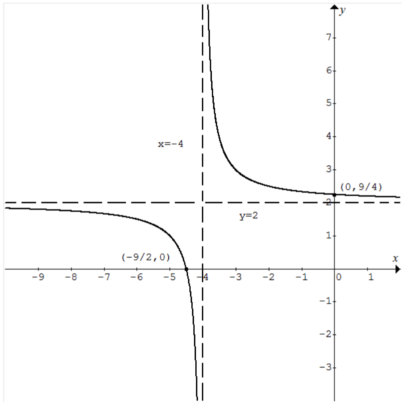
Q17ai $y=29.2-0.011\times 540\approx 23.3°\text{C}$

Q17aii For each metre increase in height above sea level, the average maximum daily temperature drops by $0.011°\text{C}$.

Q17b Latitude, because the data points are closer to the regression line. $|r|=0.897>0.494$

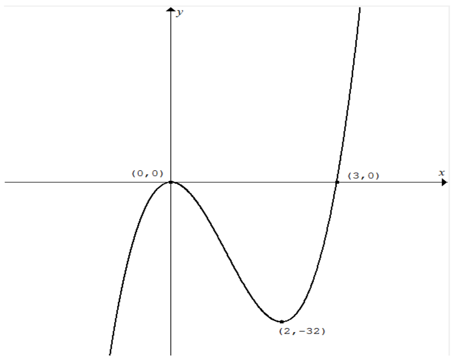
Q18 $\frac{\sin\angle ABC}{25}=\frac{\sin 28°}{16}$, $\angle ABC\approx 47°$ or $133°$
Since the angle is obtuse, $\angle ABC\approx 133°$.

Q19 Translate $y=\frac{1}{x}$ left by 4 units and up by 2 units.



Q20 Let $2\sin 4x=1$ and $0\le x\le\frac{\pi}{4}$ i.e. $0\le 4x\le\pi$
.: $\sin 4x=\frac{1}{2}$ and $4x=\frac{\pi}{6},\frac{5\pi}{6}$.: $x=\frac{\pi}{24},\frac{5\pi}{24}$

Q21 Dilate $y=f(x)$ vertically by a factor of 4 and horizontally by a factor of $\frac{1}{2}$. $(0,0)$ remains the same, $(0,6)$ becomes $(0,3)$, and $(4,-8)$ becomes $(2,-32)$.



Q22a $P(0.1<Z<0.5)=0.1915-0.0398=0.1517$

Q22b $z=\frac{3528-3300}{570}=0.4$, $P(Z>0.4)=0.5-0.1554=0.3446$
Number of babies $=1000\times 0.3446\approx 345$

Q23 $P=5000b^{\frac{-t}{10}}$ and $(20, 1250)$.: $5000b^{-2}=1250$.: $b=2$
.: $P=5000\times 2^{\frac{-t}{10}}=5000e^{\frac{-(\ln 2)t}{10}}$, $\frac{dP}{dt}=-500\ln 2\times e^{\frac{-(\ln 2)t}{10}}=-30$
$e^{\frac{-(\ln 2)t}{10}}=\frac{3}{50\ln 2}$, $\frac{-(\ln 2)t}{10}=\ln\left(\frac{3}{50\ln 2}\right)$, $t\approx 35.3$ years

Q24 Area of shaded region

$= \frac{1}{2}(2)(3) + \int_2^4 \frac{3}{x-1}\,dx = 3 + [3\ln(x-1)]_2^4 = 3 + 3\ln 3$

Q25 After the 8th deposit, amount $= 1000 \times 8.2132 = 8213.20$ dollars

Two more years, amount $8213.20 \times \left(1 + \frac{1.25}{100}\right)^2 \approx 8419.81$ dollars

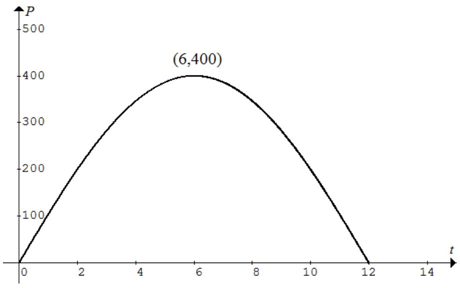
Q26a $y = -5(t^2 - 14t + 49) + 5 \times 49 + 100 = -5(t-7)^2 + 345$

Max height of 345 m when $t = 7$ s

Q26b Ground: $-5(t-7)^2 + 345 = 0$, $(t-7)^2 = 69$ ∴ $t = \sqrt{69} + 7$

$v = \frac{dy}{dt} = -10(t-7) = -10\sqrt{69}$ ms^{-1}

Q27a



Q27b $E = \int_a^b 400\sin\frac{\pi}{12}t\,dt = \left[-\frac{12}{\pi} \times 400\cos\frac{\pi}{12}t\right]_a^b$

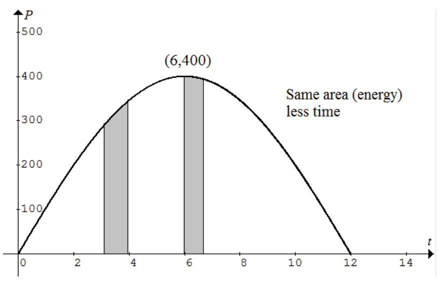
∴ $E = \frac{4800}{\pi}\left(\cos\frac{a\pi}{12} - \cos\frac{b\pi}{12}\right)$

Q27c $E = \frac{4800}{\pi}\left(\cos\frac{3\pi}{12} - \cos\frac{b\pi}{12}\right) \ge 300$

∴ $\cos\frac{b\pi}{12} \le 0.510757$, $\frac{b\pi}{12} \ge 1.03473$, $b \ge 3.95238$

Least wait time $\approx 3.95238 - 3 = 0.95238$ h $\approx 57.14 \approx 57$ minutes

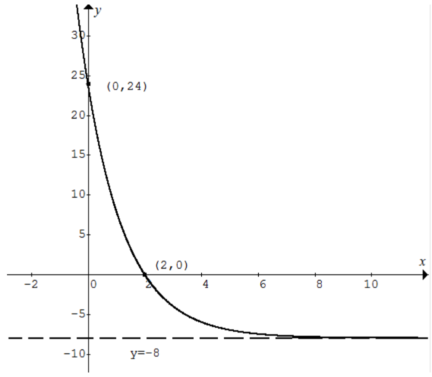
Q27d Less time (see graph below)



Q28a $\int_0^3 (8 - 2^x)\,dx = \int_0^3 \left(8 - e^{(\ln 2)x}\right)dx = \left[8x - \frac{e^{(\ln 2)x}}{\ln 2}\right]_0^3$

$= 8 \times 3 - \frac{2^3}{\ln 2} - \left(-\frac{1}{\ln 2}\right) = 24 - \frac{7}{\ln 2}$

Q28b $g(x)$ is the reflection of $f(x)$ in both axes and then translation by 5 units to the right.



Q28c $\int_2^5 g(x)\,dx = \int_2^5 -f(-(x-5))\,dx = \int_{-3}^0 -f(-u)\,du$

$= \int_3^0 f(v)\,dv = -\int_0^3 f(v)\,dv = -\left(24 - \frac{7}{\ln 2}\right) = -24 + \frac{7}{\ln 2}$

Q29a $A_0 = 5000$, $A_1 = 5000\left(1 + \frac{3}{100}\right) + 1000$, $A_2 = 1.03A_1 + 1000$

$A_3 = 1.03A_2 + 1000 \approx 8554.54$

Q29b $PV = P \times \frac{1-(1+r)^{-n}}{r}$, $30025.83 = 2000 \times \frac{1 - 1.03^{-n}}{0.03}$

∴ $1.03^n \approx 1.8195$, $n \approx \frac{\ln 1.8195}{\ln 1.03} \approx 20.25$ ∴ 20 withdrawals of \$2000

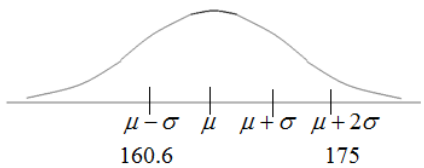
Q30 $P(X \le x) = F(x) = 1 - e^{-0.01x} = 0.99$, $e^{-0.01x} = 0.01$, $x = 460.52$

Q31 $m = \frac{dy}{dx} = 2x$, at $x = a$, $\frac{y - (a^2 - 1)}{x - a} = 2a$ ∴ $y = 2ax - a^2 - 1$

The tangents pass through $(3, -8)$ ∴ $-8 = 6a - a^2 - 1$ ∴ $a = -1, 7$

The tangents $y = 2ax - a^2 - 1$: $y = -2x - 2$, $y = 14x - 50$

Q32



Females: $\mu + 2\sigma = 175$ and $\mu - \sigma = 160.6$

∴ $\mu = 165.4$ and $\sigma = 4.8$

Males: $\mu = 1.05 \times 165.4 = 173.67$ and $\sigma = 1.1 \times 4.8 = 5.28$

Selected male:

$\Pr(H < h) = 84\%$, $h = \mu + \sigma = 173.67 + 5.28 = 178.95$ cm

Q33a $\int_0^6 \frac{Ax}{x^2+4}\,dx = 1$, $\frac{A}{2}\int_0^6 \frac{2x}{x^2+4}\,dx = 1$

Let $u = x^2 + 4$, $\frac{A}{2}\int_4^{40} \frac{1}{u}\,du = 1$, $\frac{A}{2}[\ln u]_4^{40} = 1$, $\frac{A}{2}\ln 10 = 1$, $A = \frac{2}{\ln 10}$

Q33b Let $f'(x)=\dfrac{\left(x^2+4\right)A-2Ax^2}{\left(x^2+4\right)^2}=0$, $x^2=4$.: $x=2$ is the mode

Q33c $P(X<2)=\int_0^2 \dfrac{Ax}{x^2+4}dx=\dfrac{A}{2}\left[\ln u\right]_4^8=\dfrac{A}{2}\ln 2=\dfrac{\ln 2}{\ln 10}=\log_{10}2$

Q33d $P(IQ>130)=P\left(Z>\dfrac{130-100}{15}\right)=P(Z>2)=0.025$

$P(X<2\mid IQ>130)=\dfrac{P(X<2\cap IQ>130)}{P(IQ>130)}$

$=\dfrac{P(IQ>130\cap X<2)}{P(IQ>130)}=0.80$

.: $P(IQ>130\cap X<2)=0.80\times P(IQ>130)=0.80\times 0.025=0.02$

$P(IQ>130\mid X<2)=\dfrac{P(IQ>130\cap X<2)}{P(X<2)}=\dfrac{0.02}{\log_e 2}\approx 0.066$

Q34 $S_n=1$, $\dfrac{r\left(1-r^n\right)}{1-r}=1$, $\dfrac{r-r^{n+1}}{1-r}=1$.: $r^{n+1}=2r-1$

$E(X)=r\times r^n+r^2\times r^{n-1}+r^3\times r^{n-2}+\ldots+r^n\times r=n\times r^{n+1}=n(2r-1)$

Please inform mathline@itute.com re conceptual and/or mathematical errors.

