2021 NSW ESA Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	B	D	A	B	A	D	C	A	C

Q1 $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}5\\4\end{pmatrix}$ C

Q3 $P(-2)=14$ D

Q4 $\int ydy=\int xdx$, $y^2=x^2+c$ A

Q5 $\overrightarrow{OA}.\overrightarrow{OB}=\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\theta<0$.: $\cos\theta<0$.: $\frac{\pi}{2}<\theta<\pi$ B

Q6 $P(X\geq 1)=1-P(X=0)=1-0.1^{10}>0.9$ A

Q7 $f(0)>0$, $f'(0)<0$ D

Q8 Find x and y when $t=0$ and $t=\frac{\pi}{2}$ C

Q9 Domain is R. Check y values at $x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi$ A

Q10 Distribute 3543 votes equally among the 15 candidates. Each candidate receives 236 votes. The president must receive at least two of the 3 left over votes. C

Section II

Q11ai $3\tilde{i}-\tilde{j}$

Q11b $(2a)^4-4(2a)^3b+6(2a)^2b^2-4(2a)b^3+b^4$
$=16a^4-32a^3b+24a^2b^2-8ab^3+b^4$

Q11c $\int x\sqrt{x+1}dx=\int(u-1)u^{\frac{1}{2}}du=\int\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$
$=\frac{2(x+1)^{\frac{5}{2}}}{5}-\frac{2(x+1)^{\frac{3}{2}}}{3}+c$

Q11d $\binom{10}{5}\binom{8}{3}=14112$

Q11e $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}=4\pi r^2\times 0.2$

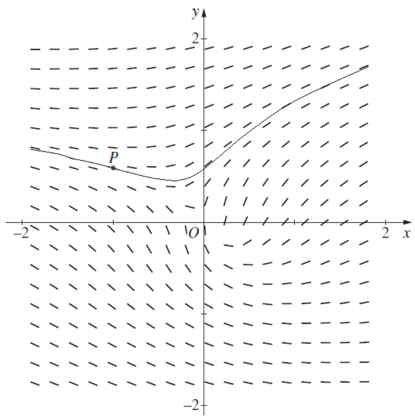
When $r=0.6$, $\frac{dV}{dt}\approx 0.9\ \text{mm}^3/\text{s}$

Q11f $\int_0^{\sqrt{3}}\frac{1}{\sqrt{4-x^2}}dx=\left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}=\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$

Q11g $2\sin^3 x+2\sin^2 x-\sin x-1=0$ and $0\leq x\leq 2\pi$
$\sin x\left(2\sin^2 x-1\right)+\left(2\sin^2 x-1\right)=0$, $(\sin x+1)\left(2\sin^2 x-1\right)=0$
.: $\sin x=-1,\pm\frac{1}{\sqrt{2}}$.: $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4}$

Q11h $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta+\alpha\beta\gamma}{\alpha\beta\gamma\delta}=\frac{-(-3)}{6}=\frac{1}{2}$

Q12a



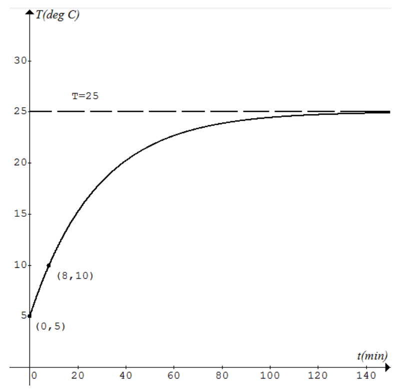
Q12bi $\frac{dt}{dT}=\frac{1}{k}.\frac{1}{T-25}$, $kt=\int\frac{1}{T-25}dT=\ln|T-25|+c$

When $t=0$, $T=5$.: $c=-\ln 20$ and $kt=\ln\frac{|T-25|}{20}$

When $t=8$, $T=10$.: $k=\frac{1}{8}\ln\frac{3}{4}$ and $t=\frac{8}{\ln\frac{3}{4}}\ln\frac{|T-25|}{20}$

When $T=20$, $t=\frac{8\ln\frac{1}{4}}{\ln\frac{3}{4}}\approx 39$ minutes

Q12bii $kt=\ln\frac{|T-25|}{20}$, $|T-25|=20e^{kt}$, $T=25-20e^{kt}$



Q12c $n=1$, $\frac{1}{1\times 2\times 3}=\frac{1}{4}-\frac{1}{2\times 2\times 3}$ is true

For $n=k$, assume

$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+...+\frac{1}{k(k+1)(k+2)}=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}$

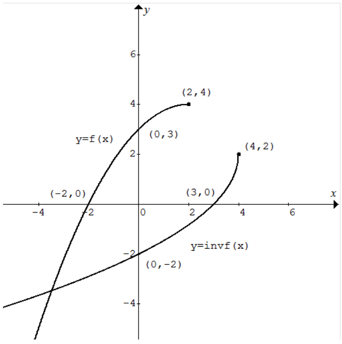
For $n=k+1$,

$\frac{1}{1\times 2\times 3}+...+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)((k+1)+1)((k+1)+2)}$
$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$
$=\frac{1}{4}-\frac{(k+3)-2}{2(k+1)(k+2)(k+3)}$
$=\frac{1}{4}-\frac{1}{2((k+1)+1)((k+1)+2)}$

.: the statement is true for all integers $n\geq 1$.

Q12di $f(x)=4-\left(1-\frac{x}{2}\right)^2=-\frac{1}{4}(x-2)^2+4$

Reflect $y=x^2$ in the x-axis, dilate vertically by a factor of $\frac{1}{4}$, translate to the right by 2 units and upwards by 4.



Q12dii Range $(-\infty, 2]$ and $x=-\frac{1}{4}(y-2)^2+4$

$\therefore y=f^{-1}(x)=2-2\sqrt{4-x}$, $x\in(-\infty, 4]$

Q12diii See part di

Q13a For $y=x^2$, $V=\int_0^2 \pi x^2 dy=\pi\int_0^2 y\,dy=\pi\left[\frac{y^2}{2}\right]_0^2=2\pi$

For $y=x^2+1$, $V=\int_1^2 \pi x^2 dy=\pi\int_1^2 (y-1)dy=\pi\left[\frac{(y-1)^2}{2}\right]_1^2=\frac{\pi}{2}$

Volume of sculpture $=2\pi-\frac{\pi}{2}=\frac{3\pi}{2}$

Q13b $x=6\sqrt{3}\,t$, $y=-5t^2+6t+1$

At the highest point, $\frac{dy}{dt}=-10t+6=0$

$\therefore t=\frac{3}{5}$ and $y=-5\left(\frac{3}{5}\right)^2+6\left(\frac{3}{5}\right)+1=\frac{14}{5}<3$, will not hit the ceiling

At the far wall, $x=6\sqrt{3}\,t=10$

$\therefore t=\frac{5}{3\sqrt{3}}$ and $y=-5\left(\frac{5}{3\sqrt{3}}\right)^2+6\left(\frac{5}{3\sqrt{3}}\right)+1\approx 2.144>0$

$\therefore$ hit the far wall without hitting the floor

Q13c Area $=2\times\int_0^2 2-x-\left(1-\frac{8}{4+x^2}\right)dx=2\times\int_0^2\left(1-x+\frac{8}{4+x^2}\right)dx$

$=2\left[x-\frac{x^2}{2}+4\tan^{-1}\frac{x}{2}\right]_0^2=2\pi$

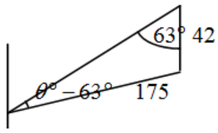
Q13di

$\frac{\sin A+\sin C}{\cos A+\cos C}=\frac{\sin(B-d)+\sin(B+d)}{\cos(B-d)+\cos(B+d)}=\frac{2\sin B\cos d}{2\cos B\cos d}=\tan B$

Q13dii Let $B=\frac{11\theta}{14}$ and $d=\frac{\theta}{14}$, $0\le\theta\le 2\pi$

$\therefore \frac{\sin\frac{5\theta}{7}+\sin\frac{6\theta}{7}}{\cos\frac{5\theta}{7}+\cos\frac{6\theta}{7}}=\tan\frac{11\theta}{14}=\sqrt{3}$ $\therefore \frac{11\theta}{14}=\frac{\pi}{3}, \frac{4\pi}{3}$ $\therefore \theta=\frac{14\pi}{33}, \frac{56\pi}{33}$

Q14a Let $\theta°$ be the bearing set for the plane.



$\frac{\sin(\theta°-63°)}{42}=\frac{\sin 63°}{175}$

$\theta°-63°\approx 12.35°$

$\therefore \theta°\approx 075°$

Q14b $\frac{dP}{dt}=0.1P\left(\frac{C-P}{C}\right)$, $0.1\frac{dt}{dP}=\frac{C}{P(C-P)}=\frac{1}{P}+\frac{1}{C-P}$

$\therefore 0.1t=\int\left(\frac{1}{P}+\frac{1}{C-P}\right)dP=\ln P-\ln(C-P)+c=\ln\frac{P}{C-P}+c$

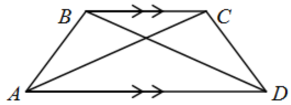
When $t=0$, $P=150000$, $0=\ln\frac{150000}{C-150000}+c$ (1)

When $t=20$, $P=600000$, $2=\ln\frac{600000}{C-600000}+c$ (2)

(2) – (1): $2=\ln\frac{4(C-150000)}{C-600000}$, $\frac{4(C-150000)}{C-600000}=e^2$, $C\approx 1130000$

Q14ci $\tilde{v}.\tilde{v}=|v||v|\cos 0=|\tilde{v}|^2$

Q14cii



$\overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}=k\tilde{b}-\tilde{a}$, $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\tilde{a}+\tilde{b}$

Since $|\overrightarrow{AC}|=|\overrightarrow{BD}|$, $\overrightarrow{AC}.\overrightarrow{BC}=\overrightarrow{BD}.\overrightarrow{BC}$, $\overrightarrow{AC}.\overrightarrow{BC}-\overrightarrow{BD}.\overrightarrow{BC}=0$

$\therefore (\overrightarrow{AC}-\overrightarrow{BD}).\overrightarrow{BC}=0$, $(\tilde{a}+\tilde{b}-(k\tilde{b}-\tilde{a})).\tilde{b}=0$

$(2\tilde{a}+(1-k)\tilde{b}).\tilde{b}=0$ Hence $2\tilde{a}.\tilde{b}+(1-k)|\tilde{b}|^2=0$

Q14d $\text{sd}(\hat{P})=\sqrt{\frac{\frac{3}{500}\left(1-\frac{3}{500}\right)}{n}}\approx\sqrt{\frac{0.005964}{n}}$, mean of $\hat{P}=\frac{3}{500}$

$P\left(\hat{P}\ge\frac{4}{500}\right)<0.025$, $P\left(Z\ge\frac{\frac{4}{500}-\frac{3}{500}}{\sqrt{\frac{0.005964}{n}}}\right)<0.025$

$\therefore \frac{\frac{4}{500}-\frac{3}{500}}{\sqrt{\frac{0.005964}{n}}}=2$, $n\approx 5964\approx 6000$

Q14e $g(x)=x^3+4x-2$, $g'(x)=3x^2+4$. At $(1,3)$, $g'(1)=7$

Note that $(g^{-1})'(3)=\frac{1}{g'(1)}=\frac{1}{7}$

Consider $f(x)=xg^{-1}(x)$, $f'(x)=g^{-1}(x)+x(g^{-1})'(x)$

At $(3,1)$, $f'(3)=g^{-1}(3)+3\times(g^{-1})'(3)=1+3\times\frac{1}{7}=\frac{10}{7}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.