2021 NSW ESA Mathematics Extension 2 Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
B	A	B	C	D	A	D	D	B	C

Q1 $\overrightarrow{AB}+\overrightarrow{CQ}=\overrightarrow{AB}+\overrightarrow{BR}=\overrightarrow{AR}=\overrightarrow{CP}$ **B**

Q2 $\frac{d}{dx}x^5e^{7x}=7x^5e^{7x}+e^{7x}5x^4$

$\therefore \frac{1}{7}x^5e^{7x}=\int x^5e^{7x}\,dx+\frac{5}{7}\int e^{7x}x^4\,dx$ **A**

Q3 $\overrightarrow{AB}=\begin{pmatrix}-2\\2\\1\end{pmatrix}-\begin{pmatrix}4\\2\\5\end{pmatrix}=-2\begin{pmatrix}3\\0\\2\end{pmatrix}$, $\tilde{r}=\begin{pmatrix}4\\2\\5\end{pmatrix}+k\overrightarrow{AB}=\begin{pmatrix}4\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\2\end{pmatrix}$ **B**

Q5 D is false when $a<0$ and $b>0$ **D**

Q6 $(x-a)(x-2-i)(x-2+i)=x^3-(4+a)x^2+(5+4a)x-5a$

$=x^3-4x^2+5x$ when $a=0$. **A**

Q7 Consider $t=0.1$, $x>-5$, $y>0$ and $z>0$. **D**

Q8 $\tilde{a}$ must have a component in the direction of $\tilde{v}$ to increase speed and a component perpendicular and to the right of the curve to change the direction of motion. **D**

Q9 It can be WIN or TRY AGAIN behind RED card, and RED or BLACK behind TRY AGAIN card. To check if the statement is true, turn over Card 3 to see the behind is not RED and, turn over Card 4 to see the behind is WIN. The validity of the statement is not affected by Card 1 whether the behind is BLACK or RED. **B**

Q10 $z=|z|\operatorname{cis}\theta$, $w=|w|\operatorname{cis}\phi$, $\frac{z}{w}=\frac{|z|}{|w|}\operatorname{cis}(\theta-\phi)$

$\operatorname{Re}\left(\frac{z}{w}\right)=\frac{|z|}{|w|}\cos(\theta-\phi)$

$\tilde{z}=|z|\cos\theta\,\tilde{i}+|z|\sin\theta\,\tilde{j}$, $\hat{w}=\cos\phi\,\tilde{i}+\sin\phi\,\tilde{j}$

Projection:

$(\tilde{z}.\hat{w})\hat{w}=|z|(\cos\theta\cos\phi+\sin\theta\sin\phi)\frac{\tilde{w}}{|w|}=\frac{|z|}{|w|}\cos(\theta-\phi)\tilde{w}$ in vector form or $\operatorname{Re}\left(\frac{z}{w}\right)w$ in complex number form. **C**

Section II

Q11a $zw=2\times 6e^{i\left(\frac{\pi}{2}+\frac{\pi}{6}\right)}=12e^{i\frac{2\pi}{3}}$

Q11b $\sum_{n=1}^{5}(i)^n=i+i^2+i^3+i^4+i^5=i-1-i+1+i=i$

Q11c $\tilde{a}.\tilde{b}=|\tilde{a}||\tilde{b}|\cos\theta$, $2=\sqrt{20}\sqrt{14}\cos\theta$, $\theta\approx 83.1°$

Q11di $-i=\operatorname{cis}\left(-\frac{\pi}{2}\right)$, one root is $(-i)^{\frac{1}{2}}=\operatorname{cis}\left(-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$,

the other root is $-\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, separated by π.

Q11dii $z^2+z+1+i=0$, $(z+1)^2=-i$, $z+1=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ or $z+1=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$ $\therefore z=-1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ or $z=-1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

Q11e $\frac{\bar{z}}{w}=\frac{(5-i)(2+4i)}{(2-4i)(2+4i)}=\frac{14+18i}{20}=\frac{7}{10}+\frac{9}{10}i$

Q11f Let $\frac{3x^2-5}{(x-2)(x^2+x+1)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+x+1}$.

$\therefore A(x^2+x+1)+(Bx+C)(x-2)=3x^2-5$

Let $x=2$, $x=0$ and $x=1$ to find $A=1$, $C=3$ and $B=2$

$\therefore \frac{3x^2-5}{(x-2)(x^2+x+1)}=\frac{1}{x-2}+\frac{2x+3}{x^2+x+1}$

Q12a $\int\frac{2x+2}{x^2+2x+2}dx+\int\frac{1}{x^2+2x+2}dx=\int\frac{1}{u}du+\int\frac{1}{1+(x+1)^2}dx$

$=\ln|x^2+2x+2|+\tan^{-1}(x+1)+c$

Q12bi If n is even, then n^2 is even.

Q12bii If n is even, then $n=2k$ where k is a positive integer, then $n^2=(2k)^2=2(2k^2)$ $\therefore n^2$ is even.

Q12c Perpendicular: $\begin{pmatrix}1\\0\\2\end{pmatrix}.\begin{pmatrix}p\\3\\-1\end{pmatrix}=0$, $p=2$

Intersect: $\tilde{r}_1=\tilde{r}_2$ $\therefore -2+\lambda=4+2\mu$, $1=-2+3\mu$ and $3+2\lambda=q-\mu$

$\therefore \mu=1$, $\lambda=8$ and $q=20$

Q12d $n=9$, $\sqrt{9!}>2^9$ is true

Assume that $\sqrt{k!}>2^k$ is true for $k>9$

For $k+1$, $\sqrt{(k+1)!}=\sqrt{k+1}\sqrt{k!}$ and $2^{k+1}=2\times 2^k$

Since $\sqrt{k+1}>2$ and $\sqrt{k!}>2^k$ $\therefore \sqrt{(k+1)!}>2^{k+1}$ is true

Hence $\sqrt{n!}>2^n$ for integers $n\geq 9$

Q12ei $\overrightarrow{HA}+\overrightarrow{HB}+\overrightarrow{HC}+\overrightarrow{HD}=\overrightarrow{HA}+\overrightarrow{HB}-\overrightarrow{HA}-\overrightarrow{HB}=\tilde{0}$

Q12eii $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}+\overrightarrow{GD}+\overrightarrow{GS}$

$=\overrightarrow{GH}+\overrightarrow{HA}+\overrightarrow{GH}+\overrightarrow{HB}+\overrightarrow{GH}+\overrightarrow{HC}+\overrightarrow{GH}+\overrightarrow{HD}+\overrightarrow{GS}$

$=\overrightarrow{GH}+\overrightarrow{GH}+\overrightarrow{GH}+\overrightarrow{GH}+\overrightarrow{GS}$

$=4\overrightarrow{GH}+\overrightarrow{GS}=\tilde{0}$

Q12eiii $\overrightarrow{HS}-\overrightarrow{HG}=\overrightarrow{GS}$ and $4\overrightarrow{GH}+\overrightarrow{GS}=\tilde{0}$

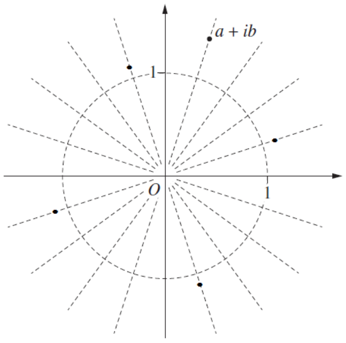
$\therefore 4\overrightarrow{GH}+\overrightarrow{HS}-\overrightarrow{HG}=-4\overrightarrow{HG}+\overrightarrow{HS}-\overrightarrow{HG}=-5\overrightarrow{HG}+\overrightarrow{HS}=\tilde{0}$

$\therefore \overrightarrow{HG}=\frac{1}{5}\overrightarrow{HS}$, $\lambda=\frac{1}{5}$

Q13a By estimation $|a+ib| \approx 1.4$, $\left|(a+ib)^{\frac{1}{4}}\right| \approx 1.09$

$\text{Arg}(a+ib) = \frac{2\pi}{5}$, $\text{Arg}(a+ib)^{\frac{1}{4}} = \frac{2\pi}{5} \times \frac{1}{4} = \frac{\pi}{10}$

The four fourth roots space out equally.



Q13b Let $u = \sqrt{x^2 - 9}$, $\frac{du}{dx} = \frac{x}{\sqrt{x^2 - 9}}$

$$\int_{\sqrt{10}}^{\sqrt{13}} x^3 \sqrt{x^2-9}\, dx = \int_{\sqrt{10}}^{\sqrt{13}} x^2 \frac{(x^2-9)x}{\sqrt{x^2-9}} dx = \int_{\sqrt{10}}^{\sqrt{13}} (u^2+9)u^2 \frac{du}{dx} dx$$

$$= \int_1^2 (u^2+9)u^2 du = \left[\frac{u^5}{5} + 3u^3\right]_1^2 = \frac{136}{5}$$

Q13ci

$$I_n = \int_1^e (\ln x)^n dx = \left[(\ln x)^n x\right]_1^e - \int_1^e n(\ln x)^{n-1} dx \quad \text{(integrate by parts)}$$

$$= (\ln e)^n e - nI_{n-1} = e - nI_{n-1} \text{ for } n \geq 1$$

Q13cii $V_A = \text{cylinder} - \frac{1}{2} \times \text{sphere} = \pi 1^2 \times 1 - \frac{1}{2} \times \frac{4}{3}\pi 1^3 = \frac{\pi}{3}$

$$V_B = \text{cylinder} - \pi\int_1^e (\ln x)^2 dx = \pi 1^2 (e-1) - \pi I_2 = \pi(e-1) - \pi(e - 2I_1)$$

Note that $I_1 = e - I_0 = e - \int_1^e 1 dx = 1$.: $V_B = \pi$

$V_A : V_B = \frac{\pi}{3} : \pi = 1:3$

Q13di $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -4(x-3)$ and $|v| = 8$ at $x = 0$

.: $v^2 = 4\left(25 - (x-3)^2\right)$, when $v = 0$, $(x-3)^2 = 25$.: $x = -2$ or 8

.: oscillates between $x = -2$ and 8.

Q13dii When $t = 0$, $x = 5.5$ and v points towards O

.: $v = \frac{dx}{dt} = -2\sqrt{25-(x-3)^2}$, $t = \frac{1}{2}\int \frac{-1}{\sqrt{25-(x-3)^2}} dx$

.: $t = \frac{1}{2}\cos^{-1}\left(\frac{x-3}{5}\right) - \frac{\pi}{6}$

When $x = 0$, first $t = \frac{1}{2}\cos^{-1}\left(-\frac{3}{5}\right) - \frac{\pi}{6} \approx 0.58$

Q14a Let $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\frac{dt}{dx} = \frac{1+t^2}{2}$

When $x = 0$, $t = 0$; when $x = \frac{\pi}{2}$, $t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{3+5\left(\frac{1-t^2}{1+t^2}\right)} dx = \int_0^{\frac{\pi}{2}} \frac{1+t^2}{3(1+t^2)+5(1-t^2)} dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{3(1+t^2)+5(1-t^2)} \frac{dt}{dx} dx = \int_0^1 \frac{1}{4-t^2} dt = \int_0^1 \frac{1}{(2+t)(2-t)} dt$$

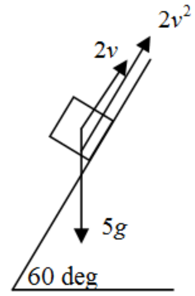
$$= \frac{1}{4}\int_0^1 \left(\frac{1}{2+t} + \frac{1}{2-t}\right) dt = \frac{1}{4}\left[\ln\left|\frac{2+t}{2-t}\right|\right]_0^1 = \frac{\ln 3}{4}$$

Q14bi

Down the slope, positive direction.

Resultant force $= 5g\sin 60° - 2v - 2v^2$

$= \frac{5\sqrt{3}}{2} g - 2v - 2v^2$ N



Q14bii Zero acceleration:

$0 = \frac{5\sqrt{3}}{2} g - 2v - 2v^2$, $2v^2 + 2v - 25\sqrt{3} = 0$ given $g = 10$

Speed $v = \frac{-2 + \sqrt{4 + 200\sqrt{3}}}{4} \approx 4.2\text{ ms}^{-1}$

Q14ci de Moivre's theorem: $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Binomial Expansion: $(\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3$

$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta(\sin^2\theta)^2$

$+ i\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)$

Equate the real parts: $\cos 5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta(\sin^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Q14cii $\left(e^{\frac{i\pi}{10}}\right)^5 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$, $\text{Re}\left(e^{\frac{i\pi}{10}}\right)^5 = 0$

.: $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

where $\cos\theta = \cos\frac{\pi}{10} \approx 0.951$.: $\cos^2\theta \approx 0.9$

$(16\cos^4\theta - 20\cos^2\theta + 5)\cos\theta = 0$.: $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

.: $\cos^2\theta = \frac{20+\sqrt{80}}{32} = \frac{5+\sqrt{5}}{8}$.: $\cos\theta = \text{Re}\left(e^{\frac{i\pi}{10}}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$

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Q15ai $(a-b)^2 \ge 0$.: $ab \le \frac{a^2+b^2}{2}$ for $a, b \ge 0$

Using $\sqrt{xy} \le \frac{x+y}{2}$, $\sqrt{abc} \le \frac{ab+c}{2} \le \frac{\frac{a^2+b^2}{2}+c}{2}$

.: $\sqrt{abc} \le \frac{a^2+b^2+2c}{4}$

Q15aii $\sqrt{abc} \le \frac{a^2+b^2+2c}{4}$, $\sqrt{abc} \le \frac{b^2+c^2+2a}{4}$,

$\sqrt{abc} \le \frac{c^2+a^2+2b}{4}$

Add the 3 inequalities and simplify, $\sqrt{abc} \le \frac{a^2+b^2+c^2+a+b+c}{6}$

Q15bi Let $n = 2m-1$ for $m = 1, 2, 3, \ldots$.: $n = 1, 3, 5, \ldots$

$t_n = \frac{n(n+1)}{2}$, $t_m = \frac{(2m-1)2m}{2} = 2m^2 - m$

.: For $n = 1, 3, 5, \ldots$, t_n is also hexagonal.

Q15bii Since $n = 2m-1$ is odd for all integers $m \ge 1$

.: $t_2, t_4, t_6, \ldots$ are not hexagonal.

If not hexagonal, then not $t_1, t_3, t_5, \ldots$

Q15ci $\ddot{x} = -g - kv^2$, $\frac{dv}{dt} = -k\left(\frac{g}{k}+v^2\right)$, $\frac{dt}{dv} = -\frac{1}{k}\cdot\frac{1}{\frac{g}{k}+v^2}$

$-kt = \int \frac{1}{\frac{g}{k}+v^2}\,dv = \sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,v + c$

$t = 0$, $v = u$.: $c = -\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,u$

At max. height $v = 0$.: $kt = \sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,u$, $t = \frac{1}{\sqrt{gk}}\tan^{-1}\sqrt{\frac{k}{g}}\,u$

Q15cii $\ddot{x} = -g - kv^2$, $\frac{1}{2}\frac{dv^2}{dx} = -k\left(\frac{g}{k}+v^2\right)$, $2\frac{dx}{dv^2} = -\frac{1}{k}\cdot\frac{1}{\frac{g}{k}+v^2}$

$-2kx = \int \frac{1}{\frac{g}{k}+v^2}\,dv^2 = \ln\left(\frac{g}{k}+v^2\right) + C$

At $x = 0$, $v = u$.: $C = -\ln\left(\frac{g}{k}+u^2\right)$.: $x = \frac{1}{2k}\ln\frac{\frac{g}{k}+u^2}{\frac{g}{k}+v^2}$

At max. height $v = 0$, $x = \frac{1}{2k}\ln\left(1+\frac{k}{g}u^2\right)$

Q15d $5^n = (2+3)^n = 2^n + \binom{n}{1}2^{n-1}3 + \binom{n}{2}2^{n-2}3^2 + \ldots + 3^n$ for $n \ge 2$.

All terms are positive .: $2^n + 3^n \ne 5^n$

Q16ai $\overrightarrow{OP}.\overrightarrow{OP} = x^2 + y^2 + z^2 = 1$

$(|x|+|y|+|z|)^2 \ge |x|^2 + |y|^2 + |z|^2 = 1$.: $|x|+|y|+|z| \ge 1$

Q16aii $\tilde{a}.\tilde{b} = |\tilde{a}||\tilde{b}|\cos\theta$ where θ is the angle between the two vectors and $-1 \le \cos\theta \le 1$.: $|\tilde{a}.\tilde{b}| \le |\tilde{a}||\tilde{b}|$

.: $|a_1b_1 + a_2b_2 + a_3b_3| \le \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$

Q16aiii Let $\tilde{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\tilde{b} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ for $x, y, z \in [-1, 1]$

From part aii, $|x+y+z| \le \sqrt{3}\sqrt{x^2+y^2+z^2} = \sqrt{3}$ for all possible values of $|x+y+z|$ including its maximum value $|x|+|y|+|z|$

.: $|x|+|y|+|z| \le \sqrt{3}$

Q16b $\tilde{r} = ut\cos\theta\,\tilde{i} + \left(ut\sin\theta - \frac{1}{2}gt^2\right)\tilde{j}$

$\tilde{v} = u\cos\theta\,\tilde{i} + (u\sin\theta - gt)\tilde{j}$

$\tilde{r}.\tilde{v} = 0$, $(u^2\cos^2\theta)t + \left(ut\sin\theta - \frac{1}{2}gt^2\right)(u\sin\theta - gt) = 0$

Simplify to $\frac{t}{2}\left(g^2t^2 - (3gu\sin\theta)t + 2u^2\right) = 0$

During the time of flight, $t > 0$.: $g^2t^2 - (3gu\sin\theta)t + 2u^2 = 0$

For two solutions of t, $\Delta > 0$.: $9g^2u^2\sin^2\theta - 8g^2u^2 > 0$

.: $\sin^2\theta > \frac{8}{9}$ where $0 < \theta < \frac{\pi}{2}$, $\sin\theta > \frac{2\sqrt{2}}{3}$, $\theta > \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

.: $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) < \theta < \frac{\pi}{2}$ and $t = \frac{3u\sin\theta \pm u\sqrt{9\sin^2\theta - 8}}{4g}$

Time of flight: Let $ut\sin\theta - \frac{1}{2}gt^2 = 0$, $t\left(u\sin\theta - \frac{1}{2}gt\right) = 0$

.: $0 < t < \frac{2u\sin\theta}{g}$ i.e. $0 < t < \frac{8u\sin\theta}{4g}$

Since $\sin\theta > \frac{2\sqrt{2}}{3}$.: $3u\sin\theta \pm u\sqrt{9\sin^2\theta - 8} < 8u\sin\theta$

.: the two values of t satisfy $0 < t < \frac{8u\sin\theta}{4g}$.

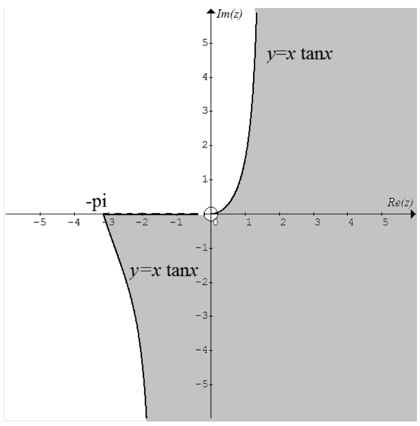
Q16c In the second quadrant, $\mathrm{Re}(z) < 0$ and $\mathrm{Arg}(z) > 0$

.: $\mathrm{Re}(z) \ge \mathrm{Arg}(z)$ is not possible.

In the fourth quadrant, $\mathrm{Re}(z) \ge \mathrm{Arg}(z)$ is true for $z \ne 0$.

In the first and third quadrant, $\mathrm{Re}(z) \ge \mathrm{Arg}(z)$ is true for some z.

$x \ge \tan^{-1}\frac{y}{x}$, $y \le x\tan x$



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