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Specialist Mathematics

2021

Trial Examination 2
(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 A block has an inertial mass of 1 kg. It slides on a rough horizontal floor and slows down uniformly. The net force (in newtons) exerted by the block on the floor **cannot** be

A. g

B. $1.1g$

C. $1.2g$

D. $1.3g$

E. $\sqrt{2}\,g$

Question 2 Vector $2\tilde{i} - 3\tilde{j}$ has two perpendicular resolutes.
One resolute is in the direction of vector $4\tilde{j} + 3\tilde{k}$. The other resolute is in the direction of vector

A. $2\tilde{i} - 3\tilde{j} + 4\tilde{k}$

B. $2\tilde{i} + 3\tilde{j} - 4\tilde{k}$

C. $-2\tilde{i} + 3\tilde{j} - 4\tilde{k}$

D. $5\tilde{i} - 2.7\tilde{j} + 3.6\tilde{k}$

E. $1.8\tilde{i} - 3.6\tilde{j} + 2.7\tilde{k}$

Question 3 For $a \in R^-$, $\left\{(x, y): y = a\sin\left(\cos^{-1}\left(\frac{x}{a}\right)\right)\right\}$ can be expressed as

A. $\left\{(x, y): y = -a\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\right\}$

B. $\left\{(x, y): y = \pm a\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\right\}$

C. $\left\{(x, y): x^2 + y^2 = a^2\right\}$

D. $\left\{(x, y): y = -\sqrt{a^2 - x^2}\right\}$

E. $\left\{(x, y): y = \sqrt{a^2 - x^2}\right\}$

Question 4 Let $z_n = \text{cis}\left(\frac{(4n+11)\pi}{22}\right)$ and $w_n = \text{cis}\left(\frac{(6n+13)\pi}{39}\right)$. $\text{Arg}(z_0) - \text{Arg}(w_0) = \frac{\pi}{6}$ when $n = 0$.

Another value of n such that $\text{Arg}(z_n) - \text{Arg}(w_n) = \frac{\pi}{6}$ is

A. 122

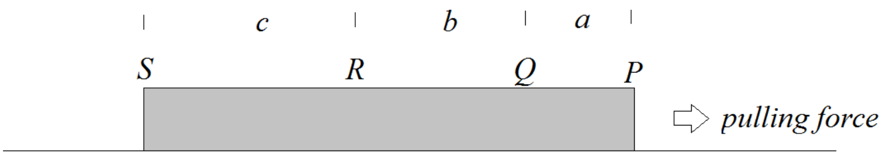
B. 139

C. 171

D. 209

E. 286

Question 5 A long heavy log *PQRS* has uniform cross-section and density.
It accelerates on a horizontal frictionless surface due to a pulling force at *P*, the front end of the log.
Lengths of the sections are: $PQ = a$, $QR = b$ and $RS = c$.
Let T_Q and T_R be the tensions in the log at *Q* and *R* respectively.



The ratio $T_R : T_Q =$

A. $c:(b+c)$

B. $b:(b+c)$

C. $a:(a+b+c)$

D. $(a+b):(b+c)$

E. $(a+b):(a+c)$

Question 6 $\tilde{a} = \tilde{i} - \tilde{j}$, $\tilde{b} = \sqrt{2}\,\tilde{j} + \tilde{k}$ and $\tilde{c}$ are linearly dependent vectors.
Vector $\tilde{c} =$

A. $\tilde{j} + \sqrt{2}\,\tilde{k}$

B. $\sqrt{2}\,\tilde{i} + 2\tilde{j}$

C. $2\tilde{i} + \sqrt{2}\,\tilde{k}$

D. $\tilde{i} - 2\tilde{j} - \sqrt{2}\,\tilde{k}$

E. $-\tilde{i} - \sqrt{2}\,\tilde{j} + \tilde{k}$

Question 7 For $a, b \in R^{+}$, the mean of the maximum and minimum values of $\left|a\,\text{cis}(\alpha)+b\,\text{cis}(\beta)\right|$ is

A. $\sqrt{a^2+b^2}$

B. b when $b>a$

C. b when $a>b$

D. a when $b>a$

E. $2(a-b)$ when $a>b$

Question 8 The real part of a possible cube root of $z=-i\,\text{cis}\left(-\frac{\pi}{4}\right)$ is

A. $\frac{\sqrt{3}+1}{2\sqrt{2}}$

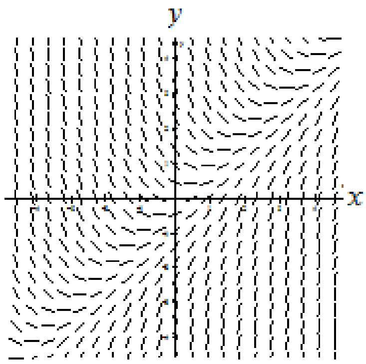
B. $\frac{\sqrt{3}-1}{2\sqrt{2}}$

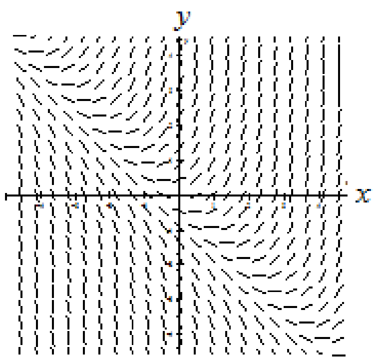
C. $\frac{1-\sqrt{2}}{2\sqrt{3}}$

D. $\frac{\sqrt{2}-1}{2\sqrt{3}}$

E. $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}$

Question 9 The graph showing the slope field of differential equation $e^{y}\frac{dx}{dy}=1$ is

A.


B.


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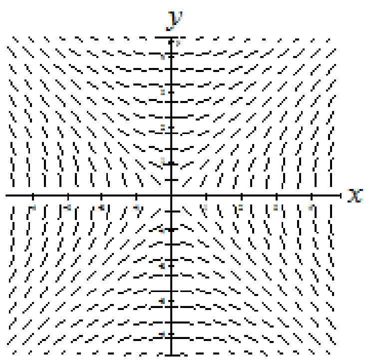
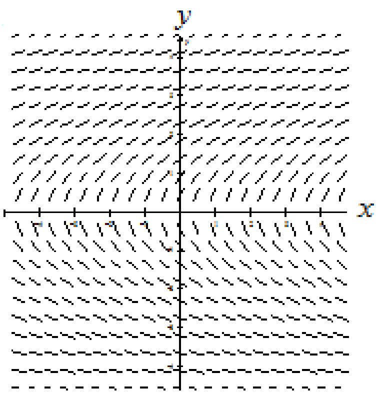
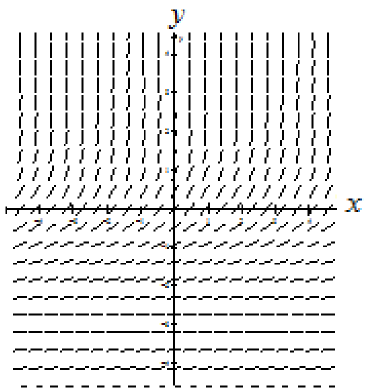


D.


E.


Question 10 Let a, $b \in R^+$, $z_1 = a + bi$, $z_2 = b + ai$, and $\theta = \text{Arg}((z_1 - i)(z_2 - 1))$. The value of $\sin\theta$ is

A. 1

B. 0.5

C. 0

D. -0.5

E. -1

Question 11 $\frac{y}{x}\frac{d}{dx}\left(\frac{x}{y}\right) + \frac{x}{y}\frac{d}{dx}\left(\frac{y}{x}\right) =$

A. xy

B. x^2y^2

C. 0

D. 1

E. 2

Question 12 A 1 kg mass has position vector $\tilde{r}(t) = (\tan t)\tilde{i} + (\sec t)\tilde{j}$, $t \geq 0$.
The change of momentum (kg m s^{-1}) in the interval $\left[0, \frac{\pi}{4}\right]$ has a magnitude of

A. $\sqrt{6} - 1$

B. $\sqrt{3} + \sqrt{2}$

C. $\sqrt{3} + 1$

D. $\sqrt{3}$

E. $\sqrt{2}$

Question 13 A particle moves in a straight line. $x(t)$, $v(t)$ and $a(t)$ are the particle's position, velocity and acceleration at time t respectively for $t \geq 0$.
Given $x(0) = x_0$, $v(t_1) = 0$, $a(t) > 0$ and $t_2 > t_1$, $x(t_2)$ is given by

A. $\int_0^{t_2} v(t)dt + x_0 - \int_{t_1}^{t_2} v(t)dt$

B. $\int_{t_1}^{t_2} v(t)dt + x_0 - \int_0^{t_1} v(t)dt$

C. $\int_0^{t_2} v(t)dt + x_0$

D. $\int_{t_1}^{t_2} v(t)dt + x_0$

E. $x_0 - \int_0^{t_2} v(t)dt$

Question 14 The graph of $f(x)=x^5-5.01x^4+10.04x^3-10.06x^2+5.04x-1$ has

A. 2 turning points and 1 point of inflection

B. 2 turning points and 2 points of inflection

C. 2 turning points and 1 stationary point of inflection

D. 1 turning point and 1 stationary point of inflection

E. 1 stationary point of inflection only

Question 15 Let $y=a\cos^{-1}\left(\frac{x}{b}-1\right)$. The value of $\int_0^{a\pi} x\,dy$ is

A. π

B. ab

C. $2ab$

D. $ab\pi$

E. $2ab\pi$

Question 16 Given $b>a$, the value of $\int_a^b \sin^4(n\theta)d\theta+\int_a^b \cos(2nt)dt-\int_a^b \cos^4(nx)dx$ is

A. $n(a-b)$

B. $n(b-a)$

C. $n(b+a)$

D. $(b-a)(b+a)$

E. 0

Question 17 For $A, B, C \in R\setminus\{0\}$, $\dfrac{x^4+x^3}{x^3-x^2-x+1}=$

A. $x+2+\dfrac{C}{(x-1)^2}+\dfrac{A}{x-1}$ for $x\neq 1$

B. $x+2+\dfrac{A}{(x-1)^2}+\dfrac{B}{x-1}$ for $x\neq \pm 1$

C. $x+2+\dfrac{B}{x-1}+\dfrac{C}{x+1}$ for $x\neq \pm 1$

D. $x+2+\dfrac{Ax+B}{(x-1)^2}+\dfrac{C}{x-1}$ for $x\neq 1$

E. $x+2+\dfrac{A}{(x-1)^2}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$ for $x\neq \pm 1$

Question 18 The weights W kg of apples from an orchard can be assumed to be normally distributed with mean of 0.20 and standard deviation of 0.01. The manager selects 16 samples, each of 25 larger apples.
Which one of the following statements is true?

A. $\mathrm{E}(\overline{W}) > 0.20$, $\mathrm{sd}(\overline{W}) < 0.0025$

B. $\mathrm{E}(\overline{W}) < 0.20$, $\mathrm{sd}(\overline{W}) = 0.0025$

C. $\mathrm{E}(\overline{W}) > 0.20$, $\mathrm{sd}(\overline{W}) < 0.002$

D. $\mathrm{E}(\overline{W}) > 0.20$, $\mathrm{sd}(\overline{W}) > 0.002$

E. $\mathrm{E}(\overline{W}) = 0.20$, $\mathrm{sd}(\overline{W}) < 0.002$

Question 19 Random variable X in a large population is normally distributed.
A random sample is taken and the values of x are: $12, 20, 21, 15, 20, 18, 15, 19, 13, 17$.
From this random sample an approximate $q\%$ confidence interval for μ of X is calculated to be $(15.7, 18.3)$.
The value of q is closest to

A. 95

B. 87

C. 86

D. 81

E. 76

Question 20 Random samples (size 50) of a particular brand of light globes were taken and the working life X of each light globe was determined.
Half of the samples had $\overline{X} > 10000$, and two thirds of the samples had $\overline{X} < 10050$.
The standard deviation of X is closest to

A. 116

B. 226

C. 527

D. 704

E. 821

SECTION B Extended-answer questions

Instructions for Section B

Answer **all** questions.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 (8 marks)

a. Find the values of a and b such that $(a+bi)^2 = -i$ and $a, b \in R$. 2 marks

b. Hence solve $z^4 + 1 = 0$ by factorisation. 2 marks

c. Sketch the solutions on an Argand diagram. 1 mark

d. Express $z^4 + 4iz^3 - 6z^2 - 4iz + 2 = 0$ in the form $(z+\alpha)^4 + \beta = 0$ where $\alpha, \beta \in C$. 2 marks

e. State a suitable transformation of $P(z) = z^4 + 4iz^3 - 6z^2 - 4iz + 2$ such that the solutions of $P(z) = 0$ form conjugate pairs.

1 mark

Question 2 (13 marks)

$P(x_1, y_1)$ is a point on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b$.

a. Sketch the graph of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Label the x and y intercepts. 2 marks

Two points $F\left(-\sqrt{a^2-b^2}, 0\right)$ and $G\left(\sqrt{a^2-b^2}, 0\right)$ are inside the ellipse.

b. Show that the length of $\overline{PF}=a+x_1\sqrt{1-\frac{b^2}{a^2}}$. 2 marks

c. Find the length of $\overline{PG}$ and then the length of $\overline{PF}+\overline{PG}$ in terms of a and b. 2 marks

d. $Q(x_2, y_2)$ is another point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

State the length of $\overline{QF}+\overline{QG}$ in terms of a and b. Explain your answer. 1 mark

e. Show that the gradient of the tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1,\ y_1)$ is $-\frac{b^2 x_1}{a^2 y_1}$. 2 marks

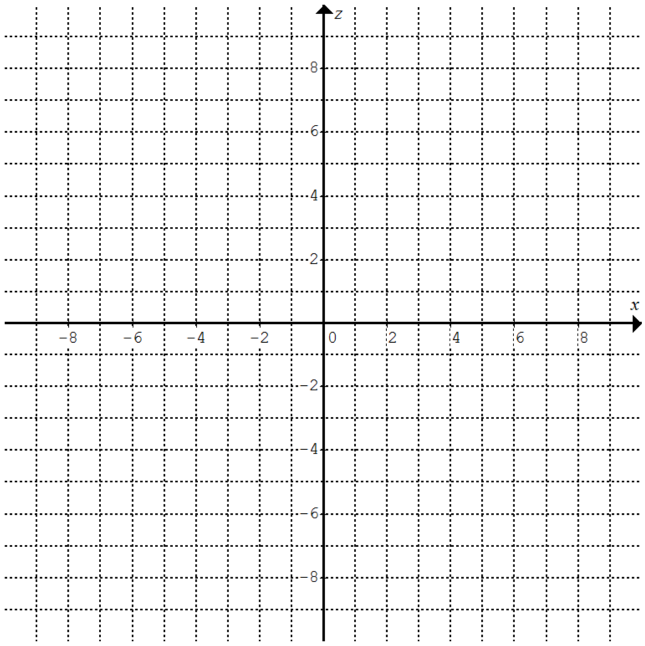
f. The tangent in part e makes angles α and β with $\overline{PF}$ and $\overline{PG}$ respectively.
Show that $\alpha=\beta$. 4 marks

Question 3 (12 marks)

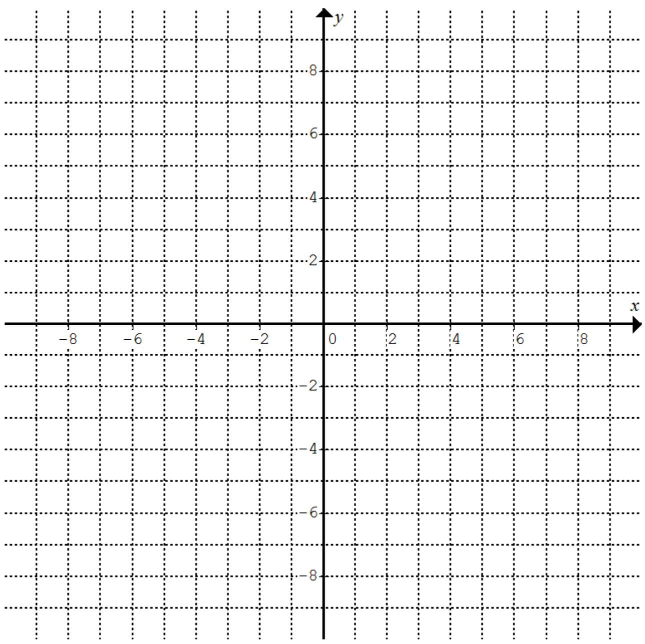
The position of a particle at time $t \geq 0$ is given by $\tilde{r}(t) = t\,\tilde{i} + \sqrt{1+t^2}\,\tilde{j} + \frac{t}{\sqrt{3}}\tilde{k}$.

Unit vectors $\tilde{i}$, $\tilde{j}$ and $\tilde{k}$ are in the directions east (x-axis), north (y-axis) and upwards (z-axis) respectively.

a. Sketch the projection of the path of the particle on the x-z plane. 2 marks



b. Sketch the projection of the path of the particle on the x-y plane. 2 marks



c. Determine the angle between the path of the particle and the horizontal plane at $t = 0$. 2 marks

d. Find the eventual speed and direction (compass bearing) of motion of the particle. 2 marks

e. Find the time (correct to 4 decimal places) taken to travel a distance of 10 units from the initial position.

Hint: Distance travelled from $t = t_1$ to $t = t_2$ is given by $\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$.

3 marks

f. Find the closest distance (correct to 4 decimal places) of the particle from the point $(2, 2, 2)$. 2 marks

Question 4 (12 marks)
A 1 kg particle is projected vertically upwards at time $t = 0$ and velocity $v = V_0$ at position $x = 0$.

It experiences a resistive force of kv^2 where k is a real constant.
Time is measured in seconds, length in metres and force in newtons.

a. Draw a labeled diagram showing the forces on the particle during its upward motion. 1 mark

b. Write down the equation of motion for the upward motion of the particle. 1 mark

c. Find the relation between x and v by solving a differential equation for the upward motion of the particle, and the maximum height reached by the particle in terms of k and V_0.

3 marks

d. Find the time taken by the particle to reach its maximum height above $x = 0$ in terms of k and V_0.

3 marks

The particle experiences the same resistive force during its downward motion.
Take $t = 0$ when the particle starts to fall.

e. Find the relation between t and v by solving a differential equation for the downward motion of the particle, and the terminal velocity reached by the particle in terms of k.

4 marks

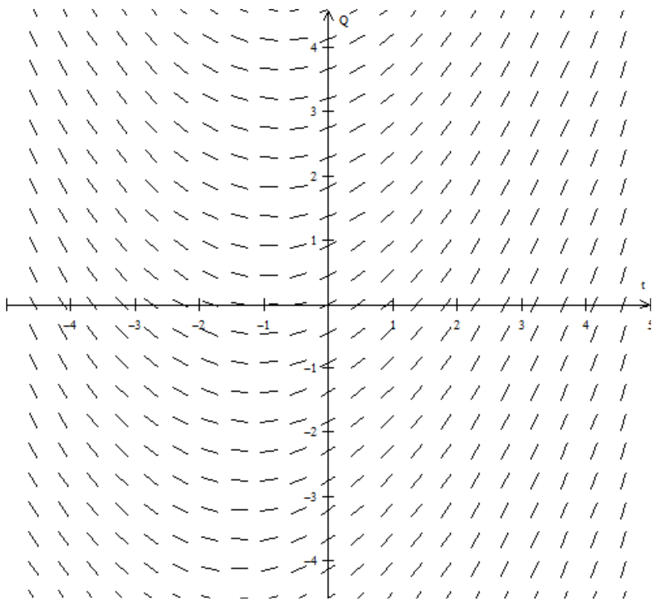
Question 5 (7 marks)

The concentration of a salt solution varies according to $\frac{1+t}{20}$ grams per litre at time $t \geq 0$ minutes.

It runs into a large tank at 10 litres per minute. The salt solution in the tank is stirred continuously.
The tank contains 100 litres of water initially and the solution runs out at 8 litres per minute.
Let Q grams be the amount of salt in the tank at time t.

a. Show that $\frac{dQ}{dt} = \frac{t^2 + 51t + 50 - 4Q}{2(50+t)}$. 2 marks

b. The slope field of the differential equation in part a is shown below.
Sketch the graph of $Q(t)$ on the slope field, and hence find Q (correct to 1 decimal place) at $t = 3$. 2 marks



c. Use Euler's method using step size of 1 to find Q (correct to 1 decimal place) at $t = 3$. 3 marks

Question 6 (8 marks)
The wait time X minutes at checkouts of a supermarket chain has a normal distribution with $\mu = 3.70$ and $\sigma = 0.60$.
At a particular store, the manager suspects that the mean wait time at the store checkouts is longer than the average of the supermarket chain.
To test this, the manager records the wait time for a random sample of 100 customers at the store and calculates the mean wait time to be 3.80 minutes.

Correct numerical answers to 2 decimal places unless stated otherwise.

a. Find the mean and the standard deviation of $\overline{X}$. 2 marks

b. Calculate the 95% confidence interval for μ. 1 mark

c. Write down the null and alternative hypotheses for the store-manager's test. 1 mark

d. Determine the *p*-value for the test, correct to 3 decimal places. 1 mark

e. Explain whether the store-manager's suspicion is justified at the 5% significance level? 1 mark

f. In the language of hypothesis testing, describe and explain the type (Type I or Type II) of error of judgment made if the manager ignored the test result and took no action.
2 marks

End of Exam 2

