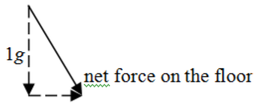
2021 Specialist Mathematics Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
A	D	D	E	A	C	B	B	E	A

11	12	13	14	15	16	17	18	19	20
C	D	C	A	D	E	B	C	D	E

Q1 A



Q2

Let $\tilde{a} = 2\tilde{i} - 3\tilde{j}$ and $\tilde{b} = 4\tilde{j} + 3\tilde{k}$, $\hat{b} = \frac{1}{5}\left(4\tilde{j} + 3\tilde{k}\right)$

$\left(\tilde{a}.\hat{b}\right)\hat{b} = -\frac{12}{25}\left(4\tilde{j} + 3\tilde{k}\right)$, $\tilde{a} - \left(\tilde{a}.\hat{b}\right)\hat{b} = 2\tilde{i} - 3\tilde{j} + \frac{12}{25}\left(4\tilde{j} + 3\tilde{k}\right)$

$= 2\tilde{i} - \frac{27}{25}\tilde{j} + \frac{36}{25}\tilde{k} = \frac{1}{2.5}\left(5\tilde{i} - 2.7\tilde{j} + 3.6\tilde{k}\right)$ D

Q3

Let $\theta = \cos^{-1}\left(\frac{x}{a}\right)$, $a < 0$ and $0 \le \theta \le \pi$

$\sin\theta = \sqrt{1 - \frac{x^2}{a^2}}$.: $a\sin\theta = a\sqrt{1 - \frac{x^2}{a^2}} = -\sqrt{a^2}\sqrt{1 - \frac{x^2}{a^2}}$

.: $a\sin\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = -\sqrt{a^2 - x^2}$ D

Q4

$\frac{(4n+11)\pi}{22} - \frac{(6n+13)\pi}{39} = 2n\pi\left(\frac{1}{11} - \frac{1}{13}\right) + \frac{\pi}{6} = \frac{2n\pi}{143} + \frac{\pi}{6}$

.: $n = 143, 286, 429, \ldots$ E

Q5

Let k be the acceleration, and the mass of a section be represented by its length.

$T_R = c \times k$, $T_Q = (b+c)k$, $\frac{T_R}{T_Q} = \frac{c}{b+c}$ A

Q6

$\tilde{c} = m\left(\tilde{i} - \tilde{j}\right) + n\left(\sqrt{2}\tilde{j} + \tilde{k}\right) = m\tilde{i} + \left(n\sqrt{2} - m\right)\tilde{j} + n\tilde{k}$

Only choice C satisfies the requirements C

Q7

$\left|a\,\text{cis}(\alpha) + b\,\text{cis}(\beta)\right|_{\max} = a + b$ when $\alpha = \beta$,

$\left|a\,\text{cis}(\alpha) + b\,\text{cis}(\beta)\right|_{\min} = |a - b|$ when $\alpha + \pi = \beta$

When $a > b$, the mean of the max and min $= a$; when $b > a$, the mean $= b$ B

Q8

$z = -i\,\text{cis}\left(-\frac{\pi}{4}\right) = \text{cis}\left(-\frac{3\pi}{4} + 2k\pi\right)$, a possible $z^{\frac{1}{3}} = \text{cis}\left(-\frac{\pi}{4} + \frac{2\pi}{3}\right)$,

$\text{Re}\left(z^{\frac{1}{3}}\right) = \cos\left(-\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right) - \sin\left(-\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$ B

Q9

$\frac{dy}{dx} = e^y$, at $y = 0$, $\frac{dy}{dx} = 1$ for all x E

Q10

$\theta = \text{Arg}\left((z_1 - i)(z_2 - 1)\right) = \text{Arg}(z_1 - i) + \text{Arg}(z_2 - 1) = \frac{\pi}{2}$, $\sin\theta = 1$ A

Q11

$\frac{y}{x}\frac{d}{dx}\left(\frac{x}{y}\right) + \frac{x}{y}\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{y}{x} \times \frac{y - x\frac{dy}{dx}}{y^2} + \frac{x}{y} \times \frac{x\frac{dy}{dx} - y}{x^2} = 0$ C

Q12

$\dot{r}(t) = \left(\sec^2 t\right)\tilde{i} + \left(\sec^2 t\,\sin t\right)\tilde{j}$, $\Delta\tilde{p} = \dot{r}\left(\frac{\pi}{4}\right) - \dot{r}(0) = \tilde{i} + \sqrt{2}\tilde{j}$

.: $|\Delta\tilde{p}| = \sqrt{3}$ D

Q13

Displacement $s = \int_0^{t_2} v(t)\,dt$, $x(t_2) = x_0 + s = \int_0^{t_2} v(t)\,dt + x_0$ C

Q14

$f'(x) = 0$ when $x = 1, 1.008$, $f''(x) = 0$ when $x = 1.006$ A

Q15

$x = b\left(1 + \cos\frac{y}{a}\right)$, $\int_0^{a\pi} b\left(1 + \cos\frac{y}{a}\right)dy = b\left[y + a\sin\frac{y}{a}\right]_0^{a\pi} = ab\pi$ D

Q16

$\int_a^b \sin^4(nx)\,dx - \int_a^b \cos^4(nx)\,dx + \int_a^b \cos(2nx)\,dx$

$= \int_a^b \left(\sin^4(nx) - \cos^4(nx)\right)dx + \int_a^b \cos(2nx)\,dx$

$= \int_a^b \left(\sin^2(nx) - \cos^2(nx)\right)dx + \int_a^b \left(\cos^2(nx) - \sin^2(nx)\right)dx = 0$ E

Q17

$\frac{x^4 + x^3}{x^3 - x^2 - x + 1} = x + 2 + \frac{(3x-2)(x+1)}{(x-1)^2(x+1)}$ where $x \ne \pm 1$

$= x + 2 + \frac{3x-2}{(x-1)^2} = x + 2 + \frac{A}{(x-1)^2} + \frac{B}{x-1}$ B

Q18

$\text{sd}\left(\overline{W}\right) = \frac{0.01}{\sqrt{25}} = 0.002$ if the samples were selected from all apples.

If the samples were from the larger apples, same sample size of 25 from a smaller population (only the larger apples) is equivalent to a larger sample size (>25) from all apples. Hence $\text{sd}\left(\overline{W}\right) < 0.002$. C

Q19

$n = 10$, $\bar{x} = \frac{15.7 + 18.3}{2} = 17$, $s_x \approx 3.127$,

$\text{normalcdf}\left(15.7, 18.3, 17, \frac{3.127}{\sqrt{10}}\right) \approx 0.81$ D

Q20

$\text{sd}\left(\overline{X}\right) = \frac{\sigma_X}{\sqrt{50}}$. Half of the samples had $\overline{X} > 10000$.:

$\text{E}\left(\overline{X}\right) = \mu_X = 10000$

$\Pr\left(Z < \frac{10050 - 10000}{\frac{\sigma_X}{\sqrt{50}}}\right) = \frac{2}{3}$.: $\frac{50}{\frac{\sigma_X}{\sqrt{50}}} \approx 0.43073$, $\sigma_X \approx 821$ E

SECTION B

Q1a $a^2 - b^2 + 2abi = -i$.: $a^2 - b^2 = 0$ and $2ab = -1$

.: $b = -a$, $2a^2 = 1$, $a^2 = \frac{1}{2}$.: $a = \pm\frac{1}{\sqrt{2}}$

When $a = \frac{1}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$; when $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$

Q1b $z^4 + 1 = (z^2 - i)(z^2 + i) = 0$

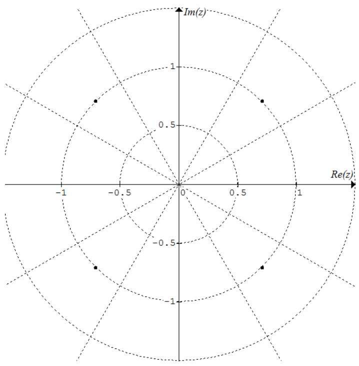
.: $z^2 = -i$, $z = a + bi = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ or $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

OR

$z^2 = i$, as in part a, $a^2 - b^2 = 0$ and $2ab = 1$

.: $a = \pm\frac{1}{\sqrt{2}}$, $b = \pm\frac{1}{\sqrt{2}}$.: $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ or $z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

Q1c



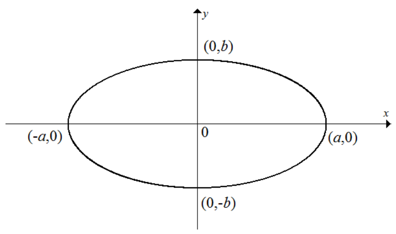
Q1d $z^4 + 4iz^3 - 6z^2 - 4iz + 2 = (z + \alpha)^4 + \beta$.: $\alpha = i$ and $\beta = 1$

.: $(z + i)^4 + 1 = 0$

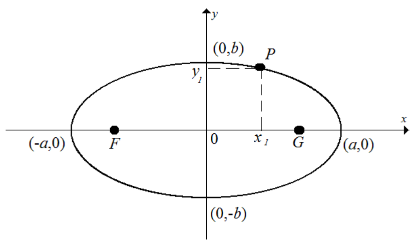
Q1e $z \to z - i$, i.e. translate 1 in the $\text{Im}(z)$ direction

.: $(z + i)^4 + 1 = 0 \to z^4 + 1 = 0$, now with real coefficients

Q2a



Q2b



$\overline{PF}^2 = \left(x_1 + \sqrt{a^2 - b^2}\right)^2 + y_1^2$ and $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

.: $\overline{PF} = a + x_1\sqrt{1 - \frac{b^2}{a^2}}$

Q2c $\overline{PG}^2 = \left(\sqrt{a^2 - b^2} - x_1\right)^2 + y_1^2$

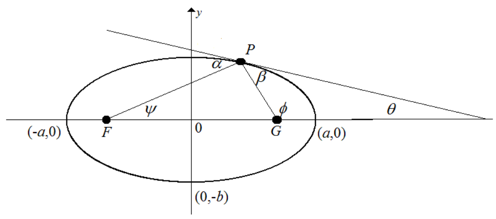
.: $\overline{PG} = a - x_1\sqrt{1 - \frac{b^2}{a^2}}$ and $\overline{PF} + \overline{PG} = 2a$

Q2d Since $\overline{PF} + \overline{PG} = 2a$, a constant for any arbitrary point P on the ellipse .: $\overline{QF} + \overline{QG} = 2a$

Q2e $\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0$, $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

.: $\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$ and at $P(x_1, y_1)$, $\frac{dy}{dx} = -\frac{b^2 x_1}{a^2 y_1}$

Q2f



$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$.: $b^2 x_1^2 + a^2 y_1^2 = a^2 b^2$

$\tan\beta = \tan((\pi - \theta) - \phi) = \frac{\tan(\pi - \theta) - \tan\phi}{1 + \tan(\pi - \theta)\tan\phi}$

$$= \frac{-\frac{b^2 x_1}{a^2 y_1} - \frac{y_1}{x_1 - \sqrt{a^2 - b^2}}}{1 + \left(-\frac{b^2 x_1}{a^2 y_1}\right)\left(\frac{y_1}{x_1 - \sqrt{a^2 - b^2}}\right)}$$

$$= \frac{b^2 x_1\sqrt{a^2 - b^2} - a^2 b^2}{(a^2 - b^2)x_1 y_1 - a^2 y_1\sqrt{a^2 - b^2}} \times \frac{(a^2 - b^2)x_1 y_1 + a^2 y_1\sqrt{a^2 - b^2}}{(a^2 - b^2)x_1 y_1 + a^2 y_1\sqrt{a^2 - b^2}}$$

Similarly

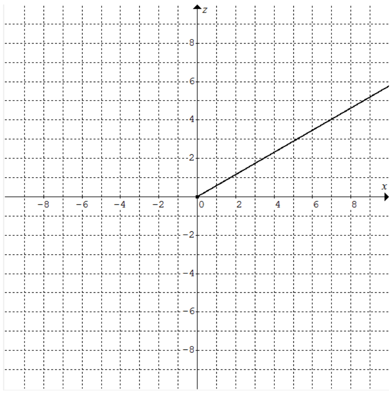
$\tan\alpha = \tan(\psi + \theta) = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\tan\theta}$

$$= \frac{\frac{y_1}{x_1 + \sqrt{a^2 - b^2}} + \frac{b^2 x_1}{a^2 y_1}}{1 - \left(\frac{y_1}{x_1 + \sqrt{a^2 - b^2}}\right)\left(\frac{b^2 x_1}{a^2 y_1}\right)}$$

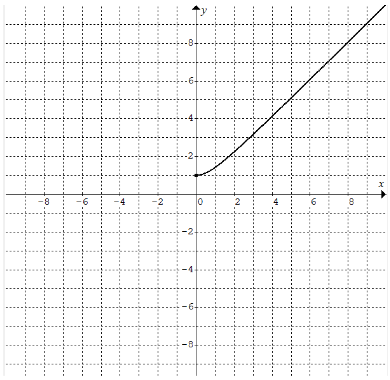
$$= \frac{b^2 x_1\sqrt{a^2 - b^2} + a^2 b^2}{(a^2 - b^2)x_1 y_1 + a^2 y_1\sqrt{a^2 - b^2}} \times \frac{(a^2 - b^2)x_1 y_1 - a^2 y_1\sqrt{a^2 - b^2}}{(a^2 - b^2)x_1 y_1 - a^2 y_1\sqrt{a^2 - b^2}}$$

Simplify to find $\tan\alpha = \tan\beta$.: $\alpha = \beta$

Q3a



Q3b



Q3c $\tan\theta = \frac{1}{\sqrt{3}}$, $\theta = \frac{\pi}{6}$

Q3d $\tilde{r}(t) = t\tilde{i} + \sqrt{1+t^2}\,\tilde{j} + \frac{t}{\sqrt{3}}\tilde{k}$, $\dot{r}(t) = \tilde{i} + \frac{t}{\sqrt{1+t^2}}\tilde{j} + \frac{1}{\sqrt{3}}\tilde{k}$

As $t \to \infty$, $\dot{r} \to \tilde{i} + \tilde{j} + \frac{1}{\sqrt{3}}\tilde{k}$

.: speed $= \sqrt{1^2 + 1^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\sqrt{21}}{3}$ in the direction NE

Q3e Let T be the time. $\int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = 10$

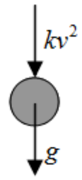
$\int_0^T \sqrt{(1)^2 + \left(\frac{t}{\sqrt{1+t^2}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2}\,dt = 10$, $T \approx 6.8740$

Q3f Distance $D = \left|(t-2)\tilde{i} + \left(\sqrt{1+t^2} - 2\right)\tilde{j} + \left(\frac{t}{\sqrt{3}} - 2\right)\tilde{k}\right|$

$= \sqrt{(t-2)^2 + \left(\sqrt{1+t^2} - 2\right)^2 + \left(\frac{t}{\sqrt{3}} - 2\right)^2}$

Minimise D to find closest distance ≈ 0.8572

Q4a



Q4b $a = \frac{F}{m}$, $a = -\left(g + kv^2\right)$

Q4c $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -\left(g + kv^2\right)$, $\frac{1}{2k}\frac{d\left(v^2\right)}{dx} = -\left(\frac{g}{k} + v^2\right)$

.: $-2kx = \int \frac{1}{\frac{g}{k} + v^2}\,d\left(v^2\right)$

$-2kx = \log_e\left(\frac{g}{k} + v^2\right) + c$ where $c = -\log_e\left(\frac{g}{k} + V_0^{\,2}\right)$

.: $-2kx = \log_e\left(\frac{\frac{g}{k} + v^2}{\frac{g}{k} + V_0^{\,2}}\right)$

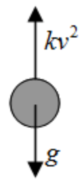
Max height is reached when $v = 0$.: $x_{\max} = \frac{1}{2k}\log_e\left(1 + \frac{k}{g}V_0^{\,2}\right)$

Q4d $\frac{dv}{dt} = -\left(g + kv^2\right) = -k\left(\frac{g}{k} + v^2\right)$, $\frac{dt}{dv} = -\frac{1}{k\left(\frac{g}{k} + v^2\right)}$

.: time taken to reach max height ($v = 0$):

$t = -\frac{1}{k}\int_{V_0}^{0} \frac{1}{\frac{g}{k} + v^2}\,dv = \frac{1}{\sqrt{gk}}\tan^{-1}\left(\sqrt{\frac{k}{g}}\,V_0\right)$

Q4e



Take downward as the positive direction.

$a = g - kv^2 = k\left(\frac{g}{k} - v^2\right)$, $\frac{dv}{dt} = k\left(\frac{g}{k} - v^2\right)$, $\frac{dt}{dk} = \frac{1}{k\left(\frac{g}{k} - v^2\right)}$

$kt = \int \frac{1}{\left(\frac{g}{k} - v^2\right)}dv = \frac{1}{2}\sqrt{\frac{k}{g}}\int\left(\frac{1}{\sqrt{\frac{g}{k}} - v} - \frac{1}{\sqrt{\frac{g}{k}} + v}\right)dv$

$= \frac{1}{2}\sqrt{\frac{k}{g}}\left(-\log_e\left(\sqrt{\frac{g}{k}} - v\right) - \log_e\left(\sqrt{\frac{g}{k}} + v\right)\right) + c$

$= -\frac{1}{2}\sqrt{\frac{k}{g}}\log_e\left(\frac{g}{k} - v^2\right) + c$

At $t = 0$, $v = 0$.: $c = \frac{1}{2}\sqrt{\frac{k}{g}}\log_e\left(\frac{g}{k}\right)$.: $t = \frac{1}{2\sqrt{gk}}\log_e\frac{\frac{g}{k}}{\frac{g}{k} - v^2}$

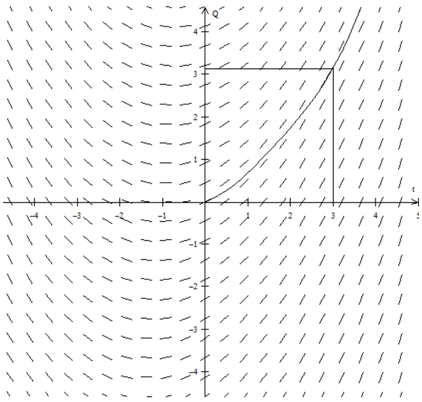
As $t \to \infty$, $\frac{g}{k} - v^2 \to 0$, $v \to \sqrt{\frac{g}{k}}$, the terminal velocity

Q5a At time t, volume of solution in the tank $V = 100 + 2t$

.: concentration of solution in the tank is $\frac{Q}{100+2t}$

.: $\frac{dQ}{dt} = \frac{1+t}{20} \times 10 - \frac{Q}{100+2t} \times 8$.: $\frac{dQ}{dt} = \frac{t^2 + 51t + 50 - 4Q}{2(50+t)}$

Q5b $Q \approx 3.2$ at $t = 3$



Q5c

$t = 0$	$Q = 0$	$\frac{dQ}{dt} \approx 0.50$
$t = 1$	$Q \approx 0 + 1 \times 0.50 \approx 0.50$	$\frac{dQ}{dt} \approx 0.98$
$t = 2$	$Q \approx 0.50 + 1 \times 0.98 \approx 1.48$	$\frac{dQ}{dt} \approx 1.44$
$t = 3$	$Q \approx 1.48 + 1 \times 1.44 \approx 2.9$	

Q6a $\mathrm{E}(\overline{X}) = \mu \approx 3.70$, $\mathrm{sd}(\overline{X}) = \frac{0.60}{\sqrt{100}} \approx 0.06$

Q6b $3.80 \pm 1.96 \times 0.06 \approx 3.68, 3.92$.: $(3.68, 3.92)$

Q6c H_0: The mean wait time at the store checkouts is the same as the average of the supermarket chain.

H_1: The mean wait time at the store checkouts is longer than the average of the supermarket chain.

Q6d $\mathrm{sd}(\overline{X}) = \frac{0.60}{\sqrt{100}} \approx 0.06$,

p-value $= \Pr(\overline{X} \ge 3.80 \mid \mu = 3.70) \approx 0.048$

Q6e Since the p-value (correct to 3 decimal places) is less than 0.05, the store-manager's suspicion is justified at 5% significance level.

Q6f The test result indicates that H_0 is not true. The manager would have made a Type II error of judgment if they ignored the test result and did not reject H_0 by not taking action.

Please inform mathline@itute.com re conceptual and mathematical errors