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2021 Specialist Mathematics

Year 12 Problem Solving Task

(Time allowed: 2.0 hours plus)

Problem Solving Task

Theme: Ropes and pulleys

Assumed knowledge:
Geometry in the plane, force and components, Newton's laws, equations, friction, calculus, use of CAS

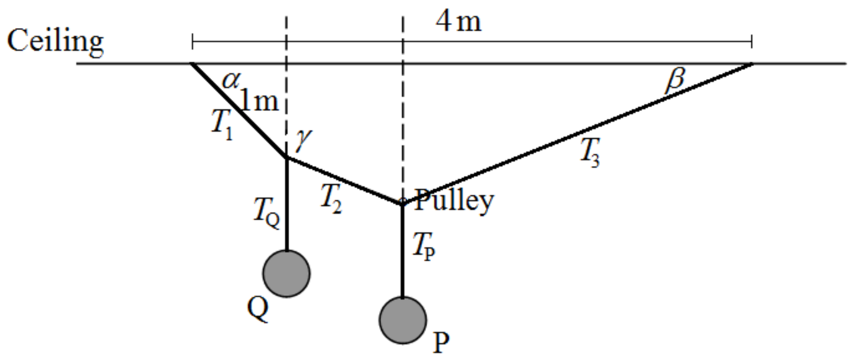
In this task, force is measured in newtons (N), distance in metres (m), time in seconds (s) and mass in kilograms (kg).

Assumptions:
(1) Ropes are taken as lines and have no mass.
(2) Pulleys are taken as points and have no mass.
(3) Pulleys are frictionless.
(4) g *is* $9.8\,N\,kg^{-1}$.

Part I (60 – 75 minutes)

A strong and non-stretchable rope is 5 m long.
Its ends are fastened to a horizontal ceiling at two points 4 m apart.
Two particles, P and Q, have the same mass of 1 kg.
Q is fastened at a point 1 m from one end of the 5 m rope, and P is attached to a pulley which runs smoothly on the remaining 4 m of the rope.

The following diagram (NOT drawn to scale) shows the two particles, ropes and pulley in **equilibrium**. T_1, T_2 and T_3 are tensions in each section of the 5 m rope. T_{Q} and T_{P} are tensions in the ropes suspending particles Q and P respectively. α, β and γ are angles (in radians) as defined in the diagram below.



a. Which one of the stated assumptions above is necessary for $T_2 = T_3$ **before** the system is in equilibrium?

b. Which one of the stated assumptions above is necessary for uniform tension in each section of the 5 m rope and in each of the ropes suspending the particles?

c. $T_{\text{P}} = T_{\text{Q}} = x$ newtons. Find x in terms of g.

d. Show that $T_2 = \dfrac{g}{2\sin\beta}$.

e. Show that $\gamma = \frac{\pi}{2} + \beta$.

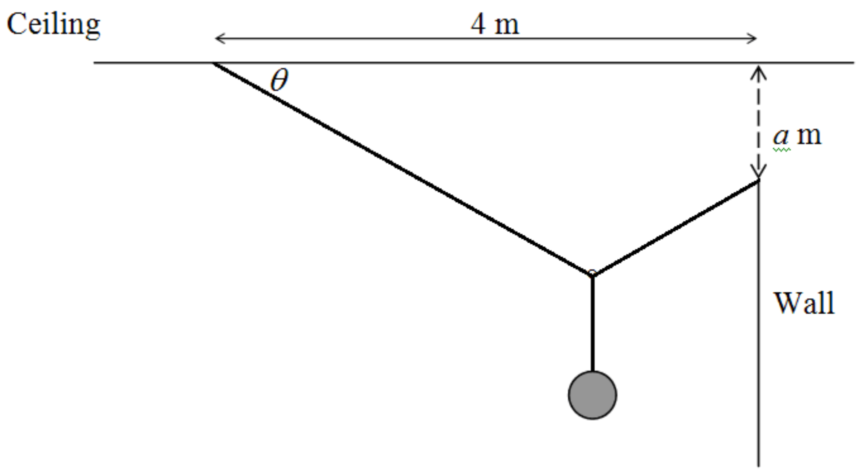
f. Show that $\tan\alpha - 3\tan\beta = 0$ and $\cos\alpha + 4\cos\beta = 4$.

g. Hence find α and β, correct to 3 decimal places.

h. Find T_1 and T_2, correct to 2 decimal places.

i. The left section of the 5 m long rope is 1 m. Find the lengths (in m) of the other two sections of the rope, correct to 3 decimal places.

Now particle Q is removed and the right end of the 5 m rope is fastened to a vertical wall a m below the ceiling as shown in the following diagram, where $a < 3$.
Particle P is still attached to the pulley which runs smoothly on the rope.
The diagram shows the system in equilibrium. The rope makes an angle of θ with the ceiling.



j. Show that the tension in the rope is $\frac{5g}{6}$ newtons.

k. Find the limiting tension in the rope as $a \to 3$. Explain.

l. If the length of the rope is ℓ m where $\ell > \sqrt{16 + a^2}$, find θ and the perpendicular distance of the pulley from the ceiling in terms of a and ℓ.

m. If the length of the rope is ℓ m where $\ell > \sqrt{16 + a^2}$, find the tension T newtons in the rope in terms of ℓ.

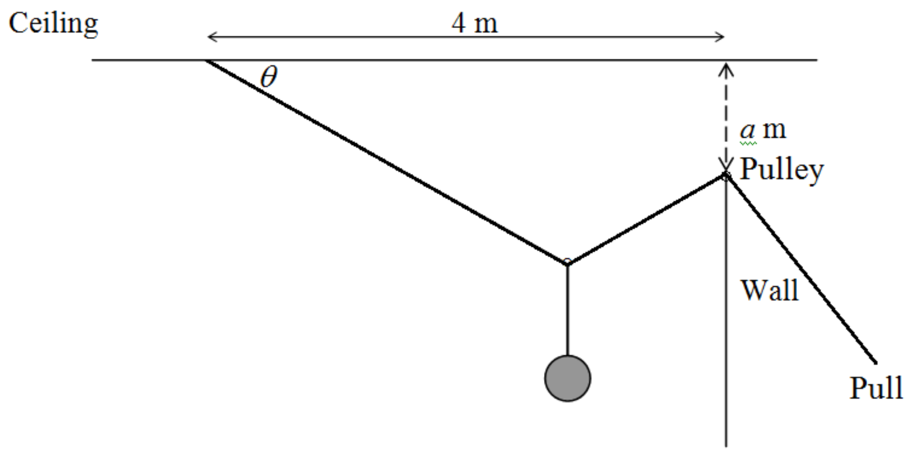
End of Part I

Part II (60 – 75 minutes)

In this task, force is measured in newtons (N), distance in metres (m), time in seconds (s) and mass in kilograms (kg).

Assumptions:
(1) Ropes are taken as lines and have no mass.
(2) Pulleys are taken as points and have no mass.
(3) Pulleys are frictionless.
(4) g is $9.8\,N\,kg^{-1}$.

A second pulley identical to the first one is placed at the top of the wall. The 5 m rope is extended and passes over the second pulley.



Initially ($t = 0$) the length of the rope on the left side of the wall is 5 m.

a. Let $a = 1$. The extended rope is pulled at the free end at $\frac{1}{10}$ m s^{-1}.
Find the exact length of the time interval during which angle θ varies.

b. Find the exact values of $\frac{d\theta}{dt}$ at the start and the end of the time interval in part a.

c. Let T newtons be the tension in the rope. Find $\frac{dT}{dt}$ in terms of θ during the time interval in part a.

Initially ($t=0$) the length of the rope on the left side of the wall is ℓ m.
The second pulley is a m below the ceiling. The extended rope is pulled at the free end at p m s^{-1}.

d. In terms of a, ℓ and p if required, find the exact length of the time interval that angle θ varies.

e. In terms of a, ℓ and p if required, find the values of $\frac{d\theta}{dt}$ at the start and the end of the interval in part d.

f. In terms of θ, a and p if required, find $\frac{dT}{dt}$ during the time interval in part d.

Initially ($t=0$) the length of the rope on the left side of the wall is ℓ m.
The second pulley is a m below the ceiling.
The extended rope is pulled at the free end with a controlled force which varies from $\frac{5g}{6}$ newtons to $\frac{g}{\sqrt{3}}$ newtons in 10 seconds at a constant rate.

g. Let T newtons be the tension in the rope. Find $\frac{dT}{dt}$.

h. Find the increase in the perpendicular distance of the particle from the ceiling in the 10 second interval.

i. Show that $\frac{d\theta}{dt} = \frac{\left(5-2\sqrt{3}\right)\sin^2\theta}{30\cos\theta}$ during the 10 second interval.

j. Using part i to verify that the time taken for the controlled force to change from $\frac{5g}{6}$ newtons to $\frac{g}{\sqrt{3}}$ newtons is indeed 10 seconds. Show working. Do not use CAS.

End of Part II

