2021 VCAA Specialist Mathematics Exam 1 Solutions

Q1a $\tilde{a} = \frac{\tilde{F}}{m} = \frac{1}{2}\tilde{i} + \frac{6}{5}\tilde{j}$

Q1b $\tilde{a}$ is constant, .: $\tilde{v} = \left(\frac{1}{2}\tilde{i} + \frac{6}{5}\tilde{j}\right)t - 3\tilde{j} = \frac{1}{2}t\,\tilde{i} + \left(\frac{6}{5}t - 3\right)\tilde{j}$

Q1c $\tilde{p}(2) = m\tilde{v}(2) = 10\tilde{i} - 6\tilde{j}$

Q2 $\int_0^1 \frac{2x}{x^2+1}dx + \int_0^1 \frac{1}{x^2+1}dx = \int_1^2 \frac{1}{u}du + \int_0^1 \frac{1}{x^2+1}dx$

$= [\log_e u]_1^2 + [\tan^{-1} x]_0^1 = \log_e 2 + \frac{\pi}{4}$

Q3a $H_0: \mu = 200$; $H_1: \mu < 200$

Q3bi $E(\bar{X}) = \mu = 200$, $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{5}{3}$

$p = \Pr(\bar{X} < 195 \mid \mu = 200) = \Pr(Z < -3) = \frac{1 - 0.9973}{2} \approx 0.001$

Q3bii 1% level, i.e. 0.01 .: $p \approx 0.001 < 0.01$, strong evidence against H_0, i.e. strong evidence that $\mu < 200$.

Q3c $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = 2$, 95% confidence interval is $(250 - 1.96\times 2,\ 250 + 1.96\times 2) \approx (246.08,\ 253.92)$.

Q4a $V_s = \int_0^\pi \pi \sin^2 x\,dx = \frac{\pi}{2}\int_0^\pi (1 - \cos 2x)\,dx = \frac{\pi^2}{2}$

Q4b The solid is dilated by a factor of $\frac{1}{k}$ in x-direction (in 1-dimension) .: $V = \frac{1}{k}V_s$.

Q5 $\frac{d}{dx}\left(e^x e^{2y} + e^{4y^2}\right) = 2e^{x+2y}\frac{dy}{dx} + e^{x+2y} + 8ye^{4y^2}\frac{dy}{dx} = 0$

At $(2,1)$, $\frac{dy}{dx} = -\frac{1}{10}$

Q6 The vectors are linearly dependent when $\tilde{c} = m\tilde{a} + n\tilde{b}$.

.: $3\tilde{i} + 2\tilde{j} + |1 - p^2|\,\tilde{k} = m\left(-\tilde{i} + 6\tilde{j} - 3\tilde{k}\right) + n\left(2\tilde{i} - 8\tilde{j} + 5\tilde{k}\right)$

$-m + 2n = 3$, $6m - 8n = 2$, $|1 - p^2| = -3m + 5n$

.: $m = 7$, $n = 5$ and $|1 - p^2| = 4$, $1 - p^2 = \pm 4$ and $p \in R$

.: $p^2 = 5$, $p = \pm\sqrt{5}$ for linearly dependent vectors

.: for the vectors to be linearly independent, $p \in R \setminus \left\{-\sqrt{5}, \sqrt{5}\right\}$

Q7a $\int \frac{1}{x}dx = \int \sin t\,dt$, $\log_e |x| = -\cos t + c$

When $t = 0$, $x = 1$.: $c = 1$ and $\log_e |x| = 1 - \cos t$

$|x| = e^{1-\cos t}$.: $x = e^{1-\cos t}$ to satisfy $(0,1)$

Q7b $x = e^{1-\cos t}$, let $\frac{dx}{dt} = e^{1-\cos t}\sin t = 0$.: $\sin t = 0$,

$t = 0, \pi, 2\pi, 3\pi, \ldots$, $x = 1$ or e^2, $x_{max} = e^2$ when $t = \pi, 3\pi, 5\pi, \ldots$

Q8a $z = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$

Q8b Let $z = a + bi$ where $a, b \in R$.

$(a + bi)^2 + 2(a - bi) + 2 = 0$, $\left(a^2 - b^2 + 2a + 2\right) + 2b(a - 1)i = 0$

.: $2b(a - 1) = 0$ and $a^2 - b^2 + 2a + 2 = 0$.: $b = 0$ or $a = 1$

When $b = 0$, $a^2 + 2a + 2 = 0$ has no real solution for a.

When $a = 1$, $b^2 = 5$.: $b = \pm\sqrt{5}$.: $z = 1 \pm \sqrt{5}\,i$

.Q9ai $x = -1 + 4\cos t$, $y = \frac{2}{\sqrt{3}}\sin t$ and $\cos^2 t + \sin^2 t = 1$

.: $\left(\frac{x+1}{4}\right)^2 + \left(\frac{\sqrt{3}\,y}{2}\right)^2 = 1$ hence $\frac{(x+1)^2}{16} + \frac{3y^2}{4} = 1$

Q9aii $3y^2 = \frac{16 - (x+1)^2}{4}$, $y^2 = \frac{-x^2 - 2x + 15}{12}$

.: $y = \frac{\sqrt{-x^2 - 2x + 15}}{2\sqrt{3}} = \frac{\sqrt{3}}{6}\sqrt{-x^2 - 2x + 15}$ in the first quadrant

Q9bi Check $\tilde{r}(t) - \tilde{s}(t) = 0$, $(4\cos t - 3\sec t)\tilde{i} + \left(\frac{2}{\sqrt{3}}\sin t - \tan t\right)\tilde{j} = 0$

$4\cos t - 3\sec t = 0$ and $\frac{2}{\sqrt{3}}\sin t - \tan t = 0$

.: $4\cos t - \frac{3}{\cos t} = 0$ and $\sin t\left(\frac{2}{\sqrt{3}} - \frac{1}{\cos t}\right) = 0$

Both equations have $\cos t = \frac{\sqrt{3}}{2}$, .: will collide at $t = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

Q9bii $\cos t = \frac{\sqrt{3}}{2}$.: $\sin t = \frac{1}{2}$.: $\tilde{r} = \tilde{s} = \left(-1 + 2\sqrt{3}\right)\tilde{i} + \frac{1}{\sqrt{3}}\tilde{j}$

.: point of collision is $\left(2\sqrt{3} - 1,\ \frac{1}{\sqrt{3}}\right)$.

Q9ci $\frac{d}{dx}\left(8\sin^{-1}\left(\frac{x+1}{4}\right) + \frac{(x+1)\sqrt{16 - (x+1)^2}}{2}\right)$

$= \frac{2}{\sqrt{1 - \left(\frac{x+1}{4}\right)^2}} + \frac{(x+1)}{2} \times \frac{-(x+1)}{\sqrt{16 - (x+1)^2}} + \frac{1}{2}\sqrt{16 - (x+1)^2}$

$= \frac{8}{\sqrt{16 - (x+1)^2}} - \frac{(x+1)^2}{2\sqrt{16 - (x+1)^2}} + \frac{\sqrt{16 - (x+1)^2}}{2}$

$= \frac{16 - (x+1)^2}{2\sqrt{16 - (x+1)^2}} + \frac{\sqrt{16 - (x+1)^2}}{2} = \sqrt{16 - (x+1)^2} = \sqrt{-x^2 - 2x + 15}$

Q9cii Area $= \int_1^{2\sqrt{3}-1} \frac{\sqrt{3}}{6}\sqrt{-2x^2 - 2x + 15}\,dx$

$= \frac{\sqrt{3}}{6}\left[8\sin^{-1}\left(\frac{x+1}{4}\right) + \frac{(x+1)\sqrt{16 - (x+1)^2}}{2}\right]_1^{2\sqrt{3}-1} = \frac{2\sqrt{3}\,\pi}{9}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.