# 2021 VCAA Specialist Mathematics Exam 2 Solutions

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**Use CAS whenever practical**

**SECTION A – Multiple-choice questions**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| B | E | E | A | D | A | E | C | B | D |

| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| B | D | E | D | A | B | E | C | D | C |

Q1 Graph $y=\dfrac{1}{\frac{1}{\cos 3x}+\frac{3}{2}}$ B

Q2 $-1\le \log_e bx\le 1$, $e^{-1}\le bx\le e$, $\dfrac{1}{be}\le x\le \dfrac{e}{b}$ E

Q3 $y_{\max}=\dfrac{1}{3}$ when $x=\pm\dfrac{\pi}{a},\pm\dfrac{3\pi}{a},\pm\dfrac{5\pi}{a},\cdots$ E

Q4 Let $z=1+i$, $\bar{z}=1-i$, $\dfrac{z\bar{z}}{z-\bar{z}}=-i$, $\text{Arg}\left(\dfrac{z\bar{z}}{z-\bar{z}}\right)=-\dfrac{\pi}{2}$ A

Q5 Circle centre $\left(2,\sqrt{3}\right)$, radius 1,

$\max|z|=\sqrt{2^2+\left(\sqrt{3}\right)^2}+1=\sqrt{7}+1$ D

Q6 $z^2=\cos 2\theta+i\sin 2\theta$, $\sin 2\theta=0$, $\theta=0,\pm\dfrac{\pi}{2},\pm\pi,\pm\dfrac{3\pi}{2},\cdots$ A

Q7 $A$ and $B$ define the first quadrant of $(x-1)^2+(y+1)^2=25$,

$\dfrac{1}{4}\times 2\pi\times 5=\dfrac{5\pi}{2}$ E

Q8 $x=1$, $y=2$, $y'=2\sin 1=1.6829$

$x=1.1$, $y=2+0.1\times 1.6829=2.1683$, $y'=2.1683\sin 1.1=1.9324$

$x=1.2$, $y=2.1683+0.1\times 1.9324=2.362$ C

Q9 $f'(x)=2(x-3)^3+5$, $f''(x)=6(x-3)^2\ge 0$, $f(x)$ is concave up on both sides of $x=3$ B

Q10 Zero gradient at $(-2,-4)$, positive gradient at $(-3,-4)$ D

Q11 $\tilde{a}=-\dfrac{5}{2}\tilde{i}-5\cos 30°\tilde{j}+10\tilde{j}$ B

Q12 $\dfrac{\tilde{a}.\tilde{b}}{ab}=\cos\theta$, $\dfrac{\tilde{b}.\tilde{c}}{bc}=\cos\phi$, $\cos\theta\cos\phi=\dfrac{-(x-1)^2}{2(1+x^2)}$ D

Q13 $\hat{b}=-\tilde{i}$, $(-4)(-\tilde{i})=4\tilde{i}$ E

Q14 When $x=2$, $a=v\dfrac{dv}{dx}=2(3+2x)=14$, $F=ma=70$ D

Q15 $F_2\sin 60°=6\sin 30°$, $6\cos 30°+F_1=F_2\cos 60°+8$

$\therefore F_2=2\sqrt{3}$, $F_1=8-2\sqrt{3}$ A

Q16 At $\theta°$, $a=g\sin\theta°$; $\cos\theta°=\dfrac{\sqrt{g^2-a^2}}{g}$

At $2\theta°$, acceleration $=g\sin 2\theta°=2g\sin\theta°\cos\theta°=\dfrac{2a}{g}\sqrt{g^2-a^2}$ B

Q17 $\text{E}(\vec{V})=\mu=1.26$, $\text{sd}(\vec{V})=\dfrac{0.01}{\sqrt{6}}$, $\Pr(\bar{V}\ge 1.25)\approx 0.9928$ E

Q18 $\bar{x}=\dfrac{70.2+75.8}{2}=73$, $73+1.96\times\dfrac{\sigma}{\sqrt{100}}=75.8$, $\dfrac{\bar{x}}{\sigma}\approx 5.1$ C

Q19 $25\to 30$, $25+6\to 30+7$ $\therefore$ $25m+n=30$ and $31m+n=37$

$\therefore m=\dfrac{7}{6}$ and $n=\dfrac{5}{6}$. When $X=32$, $S\approx 38$ D

Alternatively: $\text{E}(mX+n)=m\text{E}(X)+n$ and $\text{Var}(mX+n)=m^2\text{Var}(X)$

Q20 $\text{E}(T_1-T_2)=\text{E}(T_1)-\text{E}(T_2)=30-30=0$

$\text{Var}(T_1-T_2)=1^2\text{Var}(T_1)+(-1)^2\text{Var}(T_2)=25+25=50$, $\sigma=\sqrt{50}$

$\Pr(-3<(T_1-T)_2<3)\approx 0.329$ C

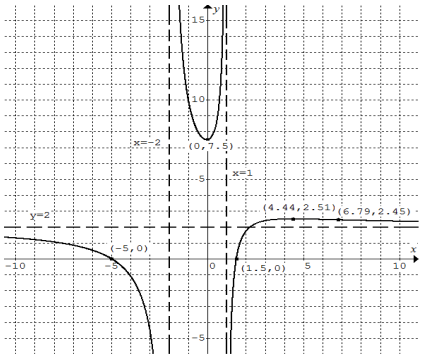
**SECTION B**

Q1a $A(x-1)(x+2)+Bx+C=(2x-3)(x+5)$, $A=2$, $B=5$,

$C=-11$ $\therefore f(x)=2+\dfrac{5x-11}{(x-1)(x+2)}$

Q1b $x=1$, $x=-2$, $y=2$

Q1c




Q1di When $(x-k)=(x+2)$, $k=-2$

When $(x-k)=(x+5)$, $k=-5$

When $2(x-k)=(2x-3)$, $k=1.5$

Q1dii Let $g_k'(x)=0$, $\therefore$ $(2k+3)x^2+(8k-30)x+(29k-30)=0$

One solution (one stationary point) when $2k+3=0$, i.e. $k=-\dfrac{3}{2}$

Two solutions when $\Delta=-21(2k^2+7k-15)>0$, i.e. $-5<k<1.5$

Given that $g_k$ has more than two asymptotes, $k\ne -5$ and $k\ne 1.5$.

No real solutions to $g_k'(x)=0$ when $\Delta=-21(2k^2+7k-15)<0$

i.e. no stationary points when $k<-5$ or $k>1.5$.

Q2ai Real coefficients, complex roots are conjugates .: $z_3 = \bar{z}_2$

Q2aii Let $z_2 = a + bi$ and $b > 0$ .: $z_3 = a - bi$

$|z_2 + z_3| = 0$ .: $a = 0$; $|z_2 - z_3| = 6$ .: $|2bi| = 6$, $b = 3$

.: $z_2 = 3i$ and $z_3 = -3i$

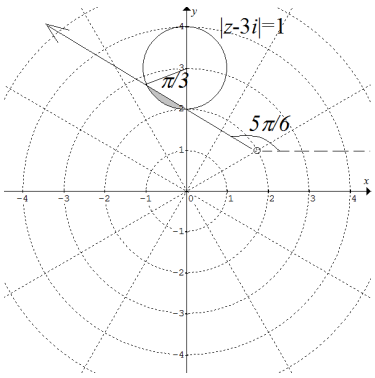
Let $z_1 = x$, $x \in R$. $p(z) = (z - x)(z - 3i)(z + 3i) = (z - x)(z^2 + 9)$

$p(2) = (2 - x)(2^2 + 9) = -13$ .: $x = 3$ and

$p(z) = (z - 3)(z^2 + 9) = z^3 - 3z^2 + 9z - 27$

Hence $\alpha = -3, \beta = 9$ and $\gamma = -27$

Q2b




Q2ci Above

Q2cii Area of the shaded region $= \frac{1}{2} \times 1^2 \left( \frac{\pi}{3} - \sin \frac{\pi}{3} \right) = \frac{2\pi - 3\sqrt{3}}{12}$

Q3ai $V = \int_0^H \pi x^2 dy = \pi \int_0^H (y + 8)^{\frac{2}{3}} dy$

Q3aii $V = \pi \left[ \frac{3(y + 8)^{\frac{5}{3}}}{5} \right]_0^H = \frac{3\pi}{5} \left( (H + 8)^{\frac{5}{3}} - 32 \right)$

Q3bi $V(h) = \frac{3\pi}{5} \left( (h + 8)^{\frac{5}{3}} - 32 \right)$, $\frac{dV}{dh} = \pi (h + 8)^{\frac{2}{3}}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, $-4\sqrt{h} = \pi (h + 8)^{\frac{2}{3}} \frac{dh}{dt}$, $\frac{dh}{dt} = -\frac{4\sqrt{h}}{\pi (h + 8)^{\frac{2}{3}}}$

Q3bii Let $\frac{d^2 h}{dt^2} = 0$, $h = 24$

Maximum rate of decrease $= \left| \frac{dh}{dt} \right| \approx 0.62$ cm/min

Q3biii When $h = 50$, leak rate $= 4\sqrt{50} = 20\sqrt{2}$

.: maximum add rate $= 20\sqrt{2}$ cm$^3$/min

Q3c $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = 40\sqrt{2} - 4\sqrt{h}$, $\frac{dh}{dt} = \frac{4\left(10\sqrt{2} - \sqrt{h}\right)}{\pi (h + 8)^{\frac{2}{3}}}$

$\frac{dt}{dh} = \frac{\pi (h + 8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)}$, $t = \int_{25}^{50} \frac{\pi (h + 8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)} dh \approx 31.4$ min

Q4a $x = ut \cos\theta$, $t = \frac{x}{u \cos\theta}$

$y = ut \sin\theta - \frac{1}{2}(9.8)t^2 = \frac{x \sin\theta}{\cos\theta} - 4.9\left( \frac{x}{u \cos\theta} \right)^2 = x \tan\theta - \frac{4.9x^2}{u^2 \cos^2\theta}$

Q4b At $\theta = 30°$, $y = x \tan 30° - \frac{4.9x^2}{u^2 \cos^2 30°} = \frac{x}{\sqrt{3}} - \frac{4.9x^2}{u^2 \left(\frac{3}{4}\right)}$

At $(16, 4)$, $4 = \frac{16}{\sqrt{3}} - \frac{4.9(16)^2}{u^2 \left(\frac{3}{4}\right)}$, $u \approx 17.87$ ms$^{-1}$ is the minimum speed

Q4c Join up smoothly at $(16, 4)$: $\frac{dy}{dx} = \tan\theta - \frac{9.8x}{u^2 \cos^2\theta} = \tan 170°$

.: $\tan\theta - \frac{156.8}{u^2 \cos^2\theta} \approx -0.176327$

Also $4 = 16 \tan\theta - \frac{4.9 \times 16^2}{u^2 \cos^2\theta}$. Solve: $\theta \approx 34°$, $u \approx 16.4$ ms$^{-1}$

Q4d $t = 0$, $s = 0$, $v = 0$

$a = \frac{60}{v}$, $v \frac{dv}{ds} = \frac{60}{v}$, $\int v^2 dv = \int 60 ds$, $\frac{v^3}{3} = 60s$, $v = (180s)^{\frac{1}{3}}$

Q4e When $v = 20$ at $B$, $s = \frac{400}{9}$

From $A$ to $W$, car accelerates: $v = (180s)^{\frac{1}{3}}$

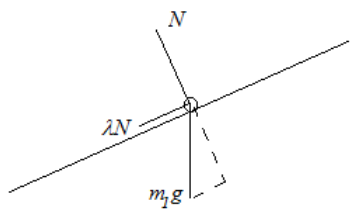
At $W$ car starts to decelerate: $v = 0$ when $s = \frac{400}{9}$

$v \frac{dv}{ds} = -9$, $v = \sqrt{800 - 18s}$

Let $(180s)^{\frac{1}{3}} = \sqrt{800 - 18s}$, $s \approx 28.082$, $WB = \frac{400}{9} - 28.082 \approx 16.4$ m

Q5a $a = 0$, $m_1 g \sin 30° - m_2 g = 0$, $m_1 = 2m_2$

Q5bi




Q5bii $N = m_1 g \cos 30° = \frac{m_1 g \sqrt{3}}{2}$, weight force down the plane $\frac{m_1 g}{2}$

Consider the two masses as a unit:

$m_2 g - \frac{\lambda m_1 g \sqrt{3}}{2} - \frac{m_1 g}{2} = (m_1 + m_2) a$, $a = \frac{\left(2m_2 - \left(\sqrt{3}\lambda + 1\right) m_1\right) g}{2(m_1 + m_2)}$

Q5c After $m_2$ hits the ground, $a = -\frac{\left(0.1\sqrt{3} + 1\right) g}{2}$, $u = 4.5$, $v = 0$

$v^2 = u^2 + 2as$, $s = 1.76$, distance is 1.76 m



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Q5d Up motion, positive: $a = -\frac{\left(0.1\sqrt{3}+1\right)g}{2}$, $u = 4.5$, $v = 0$

$v = u + at$, $t \approx 0.7828$

Down motion, positive: $a = \frac{\left(-0.1\sqrt{3}+1\right)g}{2}$, $u = 0$, $s = 1.76$

$s = ut + \frac{1}{2}at^2$, $t \approx 0.9325$ .: total time taken $\approx 1.7$ s

Q6a $\mathrm{E}\left(\overline{X}\right) = \mu = 75$, $\mathrm{sd}\left(\overline{X}\right) = \frac{\sigma}{\sqrt{n}} = \frac{8}{\sqrt{n}}$

$\Pr\left(\overline{X} > \frac{1000}{n}\right) < 0.01$, $\Pr\left(\overline{X} < \frac{1000}{n}\right) > 0.99$

$\Pr\left(Z < \frac{\frac{1000}{n} - 75}{\frac{8}{\sqrt{n}}}\right) > 0.99$, $n = 12$

Q6b $\mathrm{E}(4X) = 4\mathrm{E}(X) = 8$

$\mathrm{Var}(X_1 + X_2 + X_3 + X_4) = 4\mathrm{Var}(X) = 4 \times 0.5^2 = 1$ .: $\mathrm{sd}(X) = 1$

7.5 min to the meeting room, $\Pr(X \le 7.5) \approx 0.3085$

Q6ci $\mathrm{H}_0 : \mu = 60000$, $\mathrm{H}_1 : \mu > 60000$

Q6cii $\mathrm{E}\left(\overline{X}\right) = \mu = 60000$, $\mathrm{sd}\left(\overline{X}\right) = \frac{5000}{\sqrt{14}}$

$p = \Pr\left(\overline{X} > 63500 \mid \mu = 60000\right) \approx 0.0044$

Q6ciii $p \approx 0.0044 < 0.01$, strong evidence against $\mathrm{H}_0 : \mu = 60000$, i.e. strong evidence for the success of the advertising campaign.

Q6d $p = \Pr\left(\overline{X} \ge \overline{x} \mid \mu = 60000\right) < 0.01$, $\overline{x} \ge 63109$

Q6e $p = \Pr\left(\overline{X} \ge \overline{x} \mid \mu = 60000\right) > 0.05$ incorrectly accepted

.: $\overline{x} = 62198$ at 5% level of significance

$\Pr\left(\overline{X} \le 62198 \mid \mu = 63000\right) \approx 0.274$

*Please inform mathline@itute.com re conceptual and/or mathematical errors*


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