2021 VCAA Specialist Mathematics Exam 2 Solutions

Use CAS whenever practical

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	E	E	A	D	A	E	C	B	D

11	12	13	14	15	16	17	18	19	20
B	D	E	D	A	B	E	C	D	C

Q1 Graph $y=\dfrac{1}{\frac{1}{\cos 3x}+\frac{3}{2}}$ B

Q2 $-1\le \log_e bx\le 1$, $e^{-1}\le bx\le e$, $\dfrac{1}{be}\le x\le \dfrac{e}{b}$ E

Q3 $y_{\max}=\dfrac{1}{3}$ when $x=\pm\dfrac{\pi}{a},\pm\dfrac{3\pi}{a},\pm\dfrac{5\pi}{a},\cdots$ E

Q4 Let $z=1+i$, $\bar{z}=1-i$, $\dfrac{z\bar{z}}{z-\bar{z}}=-i$, $\text{Arg}\left(\dfrac{z\bar{z}}{z-\bar{z}}\right)=-\dfrac{\pi}{2}$ A

Q5 Circle centre $\left(2,\sqrt{3}\right)$, radius 1,

$\max|z|=\sqrt{2^2+\left(\sqrt{3}\right)^2}+1=\sqrt{7}+1$ D

Q6 $z^2=\cos 2\theta+i\sin 2\theta$, $\sin 2\theta=0$, $\theta=0,\pm\dfrac{\pi}{2},\pm\pi,\pm\dfrac{3\pi}{2},\cdots$ A

Q7 A and B define the first quadrant of $(x-1)^2+(y+1)^2=25$,

$\dfrac{1}{4}\times 2\pi\times 5=\dfrac{5\pi}{2}$ E

Q8 $x=1$, $y=2$, $y'=2\sin 1=1.6829$

$x=1.1$, $y=2+0.1\times 1.6829=2.1683$, $y'=2.1683\sin 1.1=1.9324$

$x=1.2$, $y=2.1683+0.1\times 1.9324=2.362$ C

Q9 $f'(x)=2(x-3)^3+5$, $f''(x)=6(x-3)^2\ge 0$, $f(x)$ is concave up on both sides of $x=3$ B

Q10 Zero gradient at $(-2,-4)$, positive gradient at $(-3,-4)$ D

Q11 $\tilde{a}=-\dfrac{5}{2}\tilde{i}-5\cos 30°\tilde{j}+10\tilde{j}$ B

Q12 $\dfrac{\tilde{a}.\tilde{b}}{ab}=\cos\theta$, $\dfrac{\tilde{b}.\tilde{c}}{bc}=\cos\phi$, $\cos\theta\cos\phi=\dfrac{-(x-1)^2}{2\left(1+x^2\right)}$ D

Q13 $\hat{b}=-\tilde{i}$, $(-4)\left(-\tilde{i}\right)=4\tilde{i}$ E

Q14 When $x=2$, $a=v\dfrac{dv}{dx}=2(3+2x)=14$, $F=ma=70$ D

Q15 $F_2\sin 60°=6\sin 30°$, $6\cos 30°+F_1=F_2\cos 60°+8$

$\therefore$ $F_2=2\sqrt{3}$, $F_1=8-2\sqrt{3}$ A

Q16 At $\theta°$, $a=g\sin\theta°$; $\cos\theta°=\dfrac{\sqrt{g^2-a^2}}{g}$

At $2\theta°$, acceleration $=g\sin 2\theta°=2g\sin\theta°\cos\theta°=\dfrac{2a}{g}\sqrt{g^2-a^2}$ B

Q17 $\text{E}\left(\bar{V}\right)=\mu=1.26$, $\text{sd}\left(\bar{V}\right)=\dfrac{0.01}{\sqrt{6}}$, $\Pr\left(\bar{V}\ge 1.25\right)\approx 0.9928$ E

Q18 $\bar{x}=\dfrac{70.2+75.8}{2}=73$, $73+1.96\times\dfrac{\sigma}{\sqrt{100}}=75.8$, $\dfrac{\bar{x}}{\sigma}\approx 5.1$ C

Q19 $25\to 30$, $25+6\to 30+7$ $\therefore$ $25m+n=30$ and $31m+n=37$

$\therefore$ $m=\dfrac{7}{6}$ and $n=\dfrac{5}{6}$. When $X=32$, $S\approx 38$ D

Alternatively: $\text{E}(mX+n)=m\text{E}(X)+n$ and $\text{Var}(mX+n)=m^2\text{Var}(X)$

Q20 $\text{E}(T_1-T_2)=\text{E}(T_1)-\text{E}(T_2)=30-30=0$

$\text{Var}(T_1-T_2)=1^2\,\text{Var}(T_1)+(-1)^2\,\text{Var}(T_2)=25+25=50$, $\sigma=\sqrt{50}$

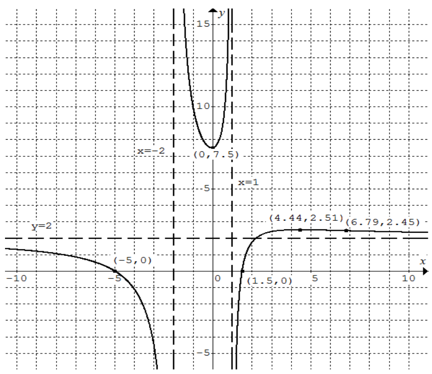
$\Pr\left(-3<(T_1-T)_2<3\right)\approx 0.329$ C

SECTION B

Q1a $A(x-1)(x+2)+Bx+C=(2x-3)(x+5)$, $A=2$, $B=5$, $C=-11$ $\therefore$ $f(x)=2+\dfrac{5x-11}{(x-1)(x+2)}$

Q1b $x=1$, $x=-2$, $y=2$

Q1c



Q1di When $(x-k)=(x+2)$, $k=-2$

When $(x-k)=(x+5)$, $k=-5$

When $2(x-k)=(2x-3)$, $k=1.5$

Q1dii Let $g_k{}'(x)=0$, solve for $x=\dfrac{-4k+15\pm\sqrt{-21\left(2k^2+7k-15\right)}}{2k+3}$

No solutions when $2k+3=0$ or $\Delta=-21\left(2k^2+7k-15\right)<0$

i.e. $k=-1.5$, $k<-5$ or $k>1.5$

Q2ai Real coefficients, complex roots are conjugates .: $z_3 = \bar{z}_2$

Q2aii Let $z_2 = a + bi$ and $b > 0$.: $z_3 = a - bi$

$|z_2 + z_3| = 0$.: $a = 0$; $|z_2 - z_3| = 6$.: $|2bi| = 6$, $b = 3$

.: $z_2 = 3i$ and $z_3 = -3i$

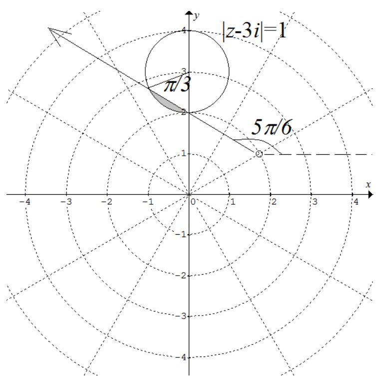
Let $z_1 = x$, $x \in R$. $p(z) = (z - x)(z - 3i)(z + 3i) = (z - x)(z^2 + 9)$

$p(2) = (2 - x)(2^2 + 9) = -13$.: $x = 3$ and

$p(z) = (z - 3)(z^2 + 9) = z^3 - 3z^2 + 9z - 27$

Hence $\alpha = -3, \beta = 9$ and $\gamma = -27$

Q2b



Q2ci Above

Q2cii Area of the shaded region $= \frac{1}{2} \times 1^2 \left(\frac{\pi}{3} - \sin\frac{\pi}{3} \right) = \frac{2\pi - 3\sqrt{3}}{12}$

Q3ai $V = \int_0^H \pi x^2 dy = \pi \int_0^H (y + 8)^{\frac{2}{3}} dy$

Q3aii $V = \pi \left[\frac{3(y+8)^{\frac{5}{3}}}{5} \right]_0^H = \frac{3\pi}{5}\left((H+8)^{\frac{5}{3}} - 32\right)$

Q3bi $V(h) = \frac{3\pi}{5}\left((h+8)^{\frac{5}{3}} - 32\right)$, $\frac{dV}{dh} = \pi(h+8)^{\frac{2}{3}}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, $-4\sqrt{h} = \pi(h+8)^{\frac{2}{3}} \frac{dh}{dt}$, $\frac{dh}{dt} = -\frac{4\sqrt{h}}{\pi(h+8)^{\frac{2}{3}}}$

Q3bii Let $\frac{d^2h}{dt^2} = 0$, $h = 24$

Maximum rate of decrease $= \left| \frac{dh}{dt} \right| \approx 0.62$ cm/min

Q3biii When $h = 50$, leak rate $= 4\sqrt{50} = 20\sqrt{2}$

.: maximum add rate $= 20\sqrt{2}$ cm^3/min

Q3c $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = 40\sqrt{2} - 4\sqrt{h}$, $\frac{dh}{dt} = \frac{4\left(10\sqrt{2} - \sqrt{h}\right)}{\pi(h+8)^{\frac{2}{3}}}$

$\frac{dt}{dh} = \frac{\pi(h+8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)}$, $t = \int_{25}^{50} \frac{\pi(h+8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)} dh \approx 31.4$ min

Q4a $x = ut\cos\theta$, $t = \frac{x}{u\cos\theta}$

$y = ut\sin\theta - \frac{1}{2}(9.8)t^2 = \frac{x\sin\theta}{\cos\theta} - 4.9\left(\frac{x}{u\cos\theta}\right)^2 = x\tan\theta - \frac{4.9x^2}{u^2\cos^2\theta}$

Q4b At $\theta = 30°$, $y = x\tan 30° - \frac{4.9x^2}{u^2\cos^2 30°} = \frac{x}{\sqrt{3}} - \frac{4.9x^2}{u^2\left(\frac{3}{4}\right)}$

At $(16, 4)$, $4 = \frac{16}{\sqrt{3}} - \frac{4.9(16)^2}{u^2\left(\frac{3}{4}\right)}$, $u \approx 17.87$ ms^{-1} is the minimum speed

Q4c Join up smoothly at $(16, 4)$: $\frac{dy}{dx} = \tan\theta - \frac{9.8x}{u^2\cos^2\theta} = \tan 170°$

.: $\tan\theta - \frac{156.8}{u^2\cos^2\theta} \approx -0.176327$

Also $4 = 16\tan\theta - \frac{4.9 \times 16^2}{u^2\cos^2\theta}$. Solve: $\theta \approx 34°$, $u \approx 16.4$ ms^{-1}

Q4d $t = 0$, $s = 0$, $v = 0$

$a = \frac{60}{v}$, $v\frac{dv}{ds} = \frac{60}{v}$, $\int v^2 dv = \int 60 ds$, $\frac{v^3}{3} = 60s$, $v = (180s)^{\frac{1}{3}}$

Q4e When $v = 20$ at B, $s = \frac{400}{9}$

From A to W, car accelerates: $v = (180s)^{\frac{1}{3}}$

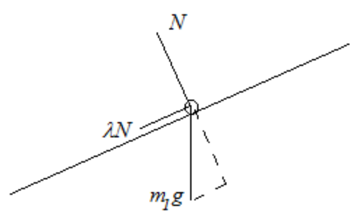
At W car starts to decelerate: $v = 0$ when $s = \frac{400}{9}$

$v\frac{dv}{ds} = -9$, $v = \sqrt{800 - 18s}$

Let $(180s)^{\frac{1}{3}} = \sqrt{800 - 18s}$, $s \approx 28.082$, $WB = \frac{400}{9} - 28.082 \approx 16.4$ m

Q5a $a = 0$, $m_1 g\sin 30° - m_2 g = 0$, $m_1 = 2m_2$

Q5bi



Q5bii $N = m_1 g\cos 30° = \frac{m_1 g\sqrt{3}}{2}$, weight force down the plane $\frac{m_1 g}{2}$

Consider the two masses as a unit:

$m_2 g - \frac{\lambda m_1 g\sqrt{3}}{2} - \frac{m_1 g}{2} = (m_1 + m_2)a$, $a = \frac{\left(2m_2 - \left(\sqrt{3}\lambda + 1\right)m_1\right)g}{2(m_1 + m_2)}$

Q5c After m_2 hits the ground, $a = -\frac{\left(0.1\sqrt{3} + 1\right)g}{2}$, $u = 4.5$, $v = 0$

$v^2 = u^2 + 2as$, $s = 1.76$, distance is 1.76 m

Q5d Up motion, positive: $a=-\frac{\left(0.1\sqrt{3}+1\right)g}{2}$, $u=4.5$, $v=0$

$v=u+at$, $t\approx 0.7828$

Down motion, positive: $a=\frac{\left(-0.1\sqrt{3}+1\right)g}{2}$, $u=0$, $s=1.76$

$s=ut+\frac{1}{2}at^2$, $t\approx 0.9325$.: total time taken ≈ 1.7 s

Q6a $\mathrm{E}\left(\overline{X}\right)=\mu=75$, $\mathrm{sd}\left(\overline{X}\right)=\frac{\sigma}{\sqrt{n}}=\frac{8}{\sqrt{n}}$

$\Pr\left(\overline{X}>\frac{1000}{n}\right)<0.01$, $\Pr\left(\overline{X}<\frac{1000}{n}\right)>0.99$

$\Pr\left(Z<\frac{\frac{1000}{n}-75}{\frac{8}{\sqrt{n}}}\right)>0.99$, $n=12$

Q6b $\mathrm{E}(4X)=4\mathrm{E}(X)=8$

$\mathrm{Var}(X_1+X_2+X_3+X_4)=4\mathrm{Var}(X)=4\times 0.5^2=1$.: $\mathrm{sd}(X)=1$

7.5 min to the meeting room, $\Pr(X\le 7.5)\approx 0.3085$

Q6ci $\mathrm{H}_0:\mu=60000$, $\mathrm{H}_1:\mu>60000$

Q6cii $\mathrm{E}\left(\overline{X}\right)=\mu=60000$, $\mathrm{sd}\left(\overline{X}\right)=\frac{5000}{\sqrt{14}}$

$p=\Pr\left(\overline{X}>63500\mid\mu=60000\right)\approx 0.0044$

Q6ciii $p\approx 0.0044<0.01$, strong evidence against $\mathrm{H}_0:\mu=60000$, i.e. strong evidence for the success of the advertising campaign.

Q6d $p=\Pr\left(\overline{X}\ge\overline{x}\mid\mu=60000\right)<0.01$, $\overline{x}\ge 63109$

Q6e $p=\Pr\left(\overline{X}\ge\overline{x}\mid\mu=60000\right)>0.05$ incorrectly accepted

.: $\overline{x}=62198$ at 5% level of significance

$\Pr\left(\overline{X}\le 62198\mid\mu=63000\right)\approx 0.274$

Please inform mathline@itute.com re conceptual and/or mathematical errors

