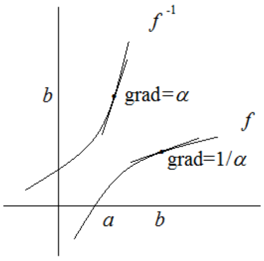
2022 Mathematical Methods Trial Exam 2 Solutions

SECTION A Multiple-choice questions

Question 1 E. $b = f^{-1}(a)$



Question 2 D. $g'(x) \neq 0$ for all $x \in R$

A differentiable periodic function have turning points where $f'(x) = 0$. The x-ccordinates of the turning points remain the same after squaring $f(x)$.

Question 3 A. $b\beta$

Horizontal dilation by a factor of b.

Question 4 A. $b^{\log_a x}$

Equation of inverse $x = a^{\log_b y}$, $\log_b y = \log_a x$, $y = b^{\log_a x}$

Question 5 E. $\left(\frac{b}{2}, a\right)$

$y = f^{-1}(x)$ and $y = g^{-1}(x)$ intersect at (b, a), $y = f^{-1}(2x)$ and $y = g^{-1}(2x)$ intersect at $\left(\frac{b}{2}, a\right)$.

Question 6 C. $f'(q) > 0$

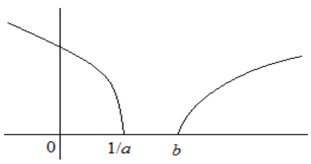
$\frac{b-a}{2} > f'(q)$ where $a > b$, .: $f'(q) < 0$

Question 7 D. $-\frac{c}{2}$

$$\int_{\left(\frac{1-a}{2}\right)}^{\left(\frac{1-b}{2}\right)} f(1-2x)\,dx = -\int_{\left(\frac{1-b}{2}\right)}^{\left(\frac{1-a}{2}\right)} f(1-2x)\,dx = -\frac{1}{2}\int_a^b f(x)dx = -\frac{1}{2}c$$

Question 8 E. $ab > 1$

$b > \frac{1}{a}$.: $ab > 1$



Question 9 D. $k = e^{-1}$

$e^{kx} = x$ and $\frac{dy}{dx} = ke^{kx} = kx = 1$.: $k = \frac{1}{x}$ and $x = e$.: $k = \frac{1}{e} = e^{-1}$

Question 10 C. $b > \frac{a}{4}$

$\frac{dy}{dx} = 4ax^3 - 2ax = 2ax(2x^2 - 1) = 0$, turning pts at $x = \pm\frac{1}{\sqrt{2}}$, $y = ax^2(x+1)(x-1) + b = -\frac{a}{4} + b > 0$.: $b > \frac{a}{4}$

Question 11 B. $3ac > b^2$

$f'(x) = 3ax^2 + 2bx + c > 0$ or < 0 .: discriminant < 0, $4b^2 - 12ac < 0$.: $3ac > b^2$

Question 12 B. $x = \frac{11\pi}{8} - b$

$\sin(2x) = \cos\left(2b - \frac{\pi}{4}\right)$, $\sin(2x) = \sin\left(\frac{\pi}{2} - \left(2b - \frac{\pi}{4}\right)\right) = \sin\left(\frac{3\pi}{4} - 2b\right)$ or $\sin\left(2\pi + \frac{3\pi}{4} - 2b\right)$

.: $x = \frac{3\pi}{8} - b$ or $\frac{11\pi}{8} - b$

Question 13 B. 2π

$f(x) = \sin(mx) + \cos(nx) = \sin(k2\pi + mx) + \cos(k2\pi + nx)$ where k is an integer.

Question 14 D. $\log_a\left(\frac{2a}{a+b}\right)$

$$\int_{a^{-1}}^{b^{-1}} \frac{b}{(bx+1)\log_e a}\,dx = \left[\frac{b}{\log_e a}\,\frac{\log_e(bx+1)}{b}\right]_{a^{-1}}^{b^{-1}}$$

$$= \left[\log_a(bx+1)\right]_{a^{-1}}^{b^{-1}} = \log_a 2 - \log_a\left(ba^{-1}+1\right) = \log_a\left(\frac{2}{ba^{-1}+1}\right) = \log_a\left(\frac{2a}{a+b}\right)$$

Question 15 C. The numbers are the same

Question 16 A. $\Pr(A') + \Pr(B')$

$\Pr\left((A \cup B)'\right) + \Pr\left((A \cap B)'\right) = \Pr(A' \cap B') + \Pr(A' \cup B')$

$= \Pr(A' \cap B') + \Pr(A') + \Pr(B') - \Pr(A' \cup B') = \Pr(A') + \Pr(B')$

Question 17 C. $\frac{1}{4}$

Trials are independent. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Question 18 C. $1 - \frac{\pi^2}{8}$

$$\int_0^{\frac{\pi}{2}} \left(a\sin(2x) + \frac{\pi}{4}\right)dx = \left[\frac{-a\cos(2x)}{2} + \frac{\pi}{4}x\right]_0^{\frac{\pi}{2}} = 1 \quad .:\ a = 1 - \frac{\pi^2}{8}$$

Question 19 A. 0.0051

$\Pr(X = 2) = 6p^2(1-p)^2 = 0.0600$.: $p \approx 0.1127$ or 0.8873

.: $\Pr(X = 3) = 4p^3(1-p) \approx 0.0511$ or 0.3149 respectively

Question 20 B. $(0.7487,\ 0.8513)$

$z = \text{invnormal}(0.80 + 0.10) \approx 1.28155$, $\text{sd}(\hat{P}) = \sqrt{\frac{0.80 \times 0.20}{100}} = 0.04$,

$\hat{p} \in (0.80 - 1.28155 \times 0.04,\ 0.80 + 1.28155 \times 0.04) \approx (0.7487,\ 0.8513)$

SECTION B

Question 1

a. $x^2=\dfrac{a\pm\sqrt{a^2-4b}}{2}$, for solutions to exist $a\pm\sqrt{a^2-4b}\ge 0$ and $a^2-4b\ge 0$

$\therefore\ a\ge\pm\sqrt{a^2-4b}$, $a^2\ge a^2-4b$ $\therefore\ b\ge 0$ and $a\ge 2\sqrt{b}$

b. $f'(x)=4x^3-2ax=0$, stationary points at $x=0,\pm\dfrac{\sqrt{a}}{2}$, a local maximum on the y-axis when $a>0$.
When $a\le 0$, there is only one stationary point (a minimum) on the y-axis.

c. $f'(x)=4x^3-2ax=0$, $x=0$ (local maximum) or $x^2=\dfrac{a}{2}$, $x=\pm\dfrac{\sqrt{a}}{2}$ (local minimum).

x-intercepts: $x^4-ax^2+1=0$, $\left(\dfrac{a}{2}\right)^2-a\left(\dfrac{a}{2}\right)+1=0$ $\therefore\ a=2$

d. For $f(x)=x^4-5x^2+4$, x-intercepts are at $x=-2,-1,1,2$
$\therefore\ g(x)=f(x-\alpha)$ satisfies the requirements when $1<-\alpha<2$ i.e. $-2<\alpha<-1$.

e. Let $x^4-4x^2+\beta=0$, $x=-\sqrt{2+\sqrt{4-\beta}}$, $-\sqrt{2-\sqrt{4-\beta}}$, $\sqrt{2-\sqrt{4-\beta}}$, $\sqrt{2+\sqrt{4-\beta}}$ arranged in ascending order. Let $\sqrt{2+\sqrt{4-\beta}}-\sqrt{2-\sqrt{4-\beta}}=\sqrt{2-\sqrt{4-\beta}}-\left(-\sqrt{2-\sqrt{4-\beta}}\right)$

$\therefore\ \sqrt{2+\sqrt{4-\beta}}=3\sqrt{2-\sqrt{4-\beta}}$, $\beta=\dfrac{36}{25}$

Question 2

a. Let $\sqrt{2x}\sin(2x)=\sqrt{2x}$ where $x\in(0,4]$. $\therefore\ \sqrt{2x}(\sin(2x)-1)=0$ $\therefore\ \sin(2x)=1$, $2x=\dfrac{\pi}{2},\dfrac{5\pi}{2}$

$\therefore\ x=\dfrac{\pi}{4},\dfrac{5\pi}{4}$ and $y=\sqrt{\dfrac{\pi}{2}},\dfrac{\sqrt{10\pi}}{2}$ respectively.

b. $y=\sqrt{2x}\sin(2x)$, $\dfrac{dy}{dx}=2\sqrt{2x}\cos(2x)+\dfrac{\sin(2x)}{\sqrt{2x}}$; $y=\sqrt{2x}$, $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{2x}}$

At $x=\dfrac{\pi}{4}$, $\dfrac{dy}{dx}=\sqrt{\dfrac{2}{\pi}}$ for both curves. At $x=\dfrac{5\pi}{4}$, $\dfrac{dy}{dx}=\sqrt{\dfrac{2}{5\pi}}$ for both curves.
$\therefore$ the curves touch but do not cross each other at the intersections.

c. Average rate of change $=\dfrac{\frac{\sqrt{10\pi}}{2}-\sqrt{\frac{\pi}{2}}}{\pi}=\dfrac{\sqrt{5}-1}{\sqrt{2\pi}}$ $\therefore\ p=5$ and $q=2\pi$

d. Area $=\displaystyle\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\left(\sqrt{2x}-\sqrt{2x}\sin(2x)\right)dx\approx 6.7678$

e. Let $\dfrac{dy}{dx}=2\sqrt{2x}\cos(2x)+\dfrac{\sin(2x)}{\sqrt{2x}}=0$ $\therefore\ 4x\cos(2x)+\sin(2x)=0$ $\therefore\ 4x+\tan(2x)=0$
$(0.9183, 1.3076)$, $(3.9585, 2.8081)$

f. $\sqrt{nx}\sin(nx)=\sqrt{nx}$, $\sqrt{nx}\sin(nx)-\sqrt{nx}=0$, $\sqrt{nx}(\sin(nx)-1)=0$ $\therefore\ \sin(nx)=1$, $nx=\dfrac{\pi}{2},\dfrac{5\pi}{2}$, $x=\dfrac{\pi}{2n},\dfrac{5\pi}{2n}$

$\therefore\ y=\sqrt{\dfrac{\pi}{n}},\sqrt{\dfrac{5\pi}{n}}$ respectively.

Gradient of line $=\dfrac{\sqrt{\frac{5\pi}{n}}-\sqrt{\frac{\pi}{n}}}{\frac{5\pi}{2n}-\frac{\pi}{2n}}=\dfrac{\sqrt{\frac{\pi}{n}}\left(\sqrt{5}-1\right)}{\frac{2\pi}{n}}=\dfrac{n}{\pi}\sqrt{\dfrac{\pi}{n}}\left(\dfrac{\sqrt{5}-1}{2}\right)=\sqrt{\dfrac{n}{\pi}}\left(\dfrac{\sqrt{5}-1}{2}\right)$ $\therefore\ k=\sqrt{\dfrac{n}{\pi}}$

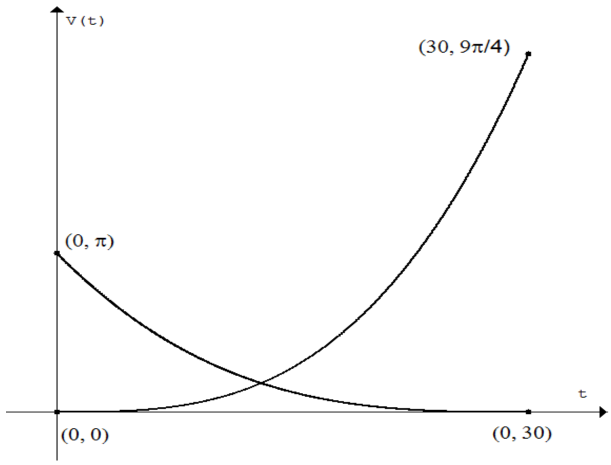
Question 3

a. Required time $= \dfrac{3}{0.1} = 30\,\text{min}$

b. $\dfrac{\text{Volume of Tank A}}{\text{Volume of Tank B}} = \dfrac{\frac{1}{3}\pi r_A^{\,2}(3)}{\frac{1}{3}\pi r_B^{\,2}(3)} = \dfrac{27000}{12000}$ $\therefore \dfrac{r_A}{r_B} = \dfrac{3}{2}$

c. When Tank A is filled from empty, its depth increases by 0.1 m min^{-1}. After 15 min, it is filled to a depth of 1.5 m. Tank B is emptied from full, its depth decreases by 0.1 m min^{-1}. After 15 min, it is emptied to a depth of $(3-1.5)=1.5$ m.

d.



e. Tank A: $t=15$, $V_A = \dfrac{\pi 15^3}{12000}$, average rate $= \dfrac{\frac{\pi 15^3}{12000} - 0}{15-0} = \dfrac{\pi 15^2}{12000} = \dfrac{9\pi}{480}$ m^3 min^{-1}

Tank B: $t=15$, $V_B = \dfrac{\pi 15^3}{27000}$, average rate $= \dfrac{\frac{\pi 15^3}{27000} - \frac{\pi 30^3}{27000}}{15-0} = \dfrac{\pi 15^2 - 2\pi 30^2}{27000} = -\dfrac{7\pi}{120}$ m^3 min^{-1}

f. $\dfrac{\pi(30-t)^3}{27000} = \dfrac{\pi t^3}{12000}$, $\left(\dfrac{30-t}{t}\right)^3 = \dfrac{27000}{12000}$, $\dfrac{30}{t} - 1 = \left(\dfrac{3}{2}\right)^{\frac{2}{3}}$, $t = \dfrac{30}{1+\left(\frac{3}{2}\right)^{\frac{2}{3}}}$ $\therefore a = \dfrac{3}{2}$

g. $\dfrac{dV_A}{dt} = \dfrac{\pi t^2}{4000}$, $\dfrac{dV_B}{dt} = -\dfrac{\pi(30-t)^2}{9000}$. Let $\dfrac{\pi(30-t)^2}{9000} = \dfrac{\pi t^2}{4000}$ $\therefore t = 12$

Question 4

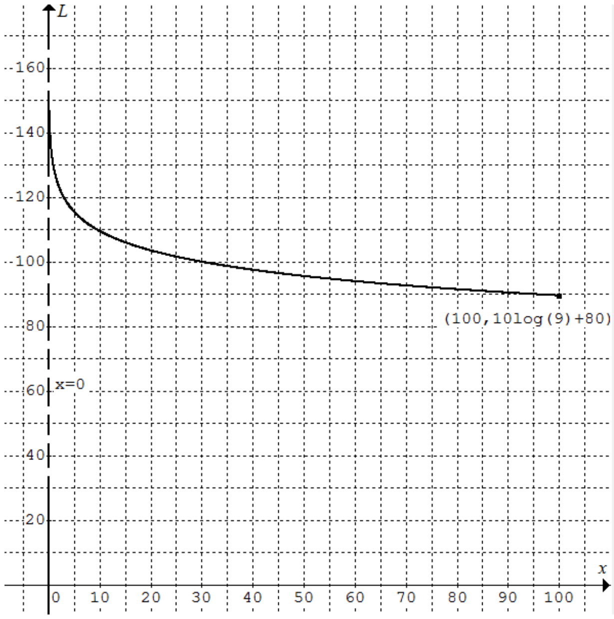
a i. $L = 10\log_{10}\left(\dfrac{k}{x^2}\right)$, $100 = 10\log_{10}\left(\dfrac{k}{30^2}\right)$, $\dfrac{k}{900} = 10^{10}$, $k = 9\times 10^{12}$

a ii. $L = 10\log_{10}\left(\dfrac{9\times 10^{12}}{100^2}\right) = 10\log_{10}\left(9\times 10^8\right) = 10(\log_{10}9 + 8) = 10\log_{10}9 + 80$

a iii. $n = 10\log_{10}\left(\dfrac{9\times 10^{12}}{d^2}\right)$, $10^{\frac{n}{10}} = \dfrac{9\times 10^{12}}{d^2}$, $d^2 = 9\times 10^{12-\frac{n}{10}}$, $d = 3\times 10^{6-\frac{n}{20}}$

a iv. $L_2 - L_1 = 10\log_{10}\left(\dfrac{k}{x_2^{\,2}}\right) - 10\log_{10}\left(\dfrac{k}{x_1^{\,2}}\right)$, $-6 = 10\log_{10}\left(\dfrac{x_1}{x_2}\right)^2$, $-\dfrac{3}{10} = \log_{10}\left(\dfrac{x_1}{x_2}\right)$, $\dfrac{x_1}{x_2} = 10^{-\frac{3}{10}}$ $\therefore \dfrac{x_2}{x_1} = 10^{\frac{3}{10}}$

a v.



a vi. $\Delta L = L_2 - L_1 = 10\log_{10}\left(\dfrac{1.8\times10^{13}}{x^2}\right) - 10\log_{10}\left(\dfrac{9\times10^{12}}{x^2}\right) = 10\log_{10} 2$

b. $L = 10\log_{10}\left(\dfrac{k}{x^2}\right) = \dfrac{10}{\log_e 10}\log_e\left(\frac{k}{x^2}\right) = \dfrac{10}{\log_e 10}(\log_e k - 2\log_e x)$, $\dfrac{dL}{dx} = \dfrac{10}{\log_e 10}\left(-\dfrac{2}{x}\right) = -\dfrac{20}{x\log_e 10}$

At $x = \dfrac{10}{\log_e 20}$, $\dfrac{dL}{dx} = -\dfrac{20}{\frac{10}{\log_e 20}\log_e 10} = -\dfrac{2\log_e 20}{\log_e 10} = -2\log_{10} 20$ or $-2(1+\log_{10} 2)$.

Question 5

a. $\hat{p} = 15\% = 0.15$, standard deviation (standard error) $= \sqrt{\dfrac{0.15\times0.85}{100}} \approx 0.0357$

$z = \text{invNorm}(0.95+0.025) \approx 1.9600$

.: $a = 0.15 - 1.9600\times0.0357 \approx 0.08$ and $a = 0.15 + 1.9600\times0.0357 \approx 0.22$

b. $\Pr(W > 1.6) = 0.15$.: $\Pr\left(z < \dfrac{1.6-\mu}{\sigma}\right) = 0.85$.: $\dfrac{1.6-\mu}{\sigma} \approx 1.03643$

$\Pr(W < 1.25) = 0.10$.: $\Pr\left(z < \dfrac{1.25-\mu}{\sigma}\right) = 0.10$.: $\dfrac{1.25-\mu}{\sigma} \approx -1.28155$

.: $\mu \approx 1.4435 \approx 1.44$ and $\sigma \approx 0.1510 \approx 0.15$

c. $\Pr(W > 1.5 \mid W > 1.25) = \dfrac{\Pr(W > 1.5)}{\Pr(W > 1.25)} \approx 0.39$

d. Binomial distribution: $n = 100$, $p = 1 - 0.15 - 0.10 = 0.75$

$\Pr(X > 75) = 1 - \text{binomcdf}(100, 0.75, 75) \approx 0.4617 \approx 0.46$

e. $\bar{x} = np = 75.00$, $\text{sd}(X) = \sqrt{np(1-p)} \approx 4.33013 \approx 4.33$

f. $\Pr(X > 75) = \text{nomalcdf}(75, E99, 75, 4.33013) \approx 0.50$

g. The sample size (100) is small comparatively.
Increasing the size of the sample decreases the difference between the two answers.

