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2022

Mathematical Methods

Trial Examination 2
(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 The gradient of the tangent to $f^{-1}(x)$ at $x=a$ is α. The gradient of the tangent to $f(x)$ at $x=b$ is $\dfrac{1}{\alpha}$. Which one of the following statements is true?

A. $b=f(a)$

B. $a=f^{-1}(b)$

C. $a-b=0$

D. $a=-f(b)$

E. $b=f^{-1}(a)$

Question 2 For $x\in R$, $f(x)$ is a **differentiable periodic odd** function of period α.
Let $g(x)=(f(x))^2$. Which of the following statements is false?

A. $g'(-x)=-g'(x)$

B. $g(x+\alpha)=g(x-\alpha)$

C. $g(x-2\alpha)=g(x)$

D. $g'(x)\neq 0$ for all $x\in R$

E. $g'(x+\alpha)=g'(x-\alpha)$

Question 3 $y=f(x)$ has exactly two *x*-intercepts separated by a distance of β.
$y=af\left(\dfrac{x-c}{b}\right)$ also has exactly two *x*-intercepts. The distance between these two *x*-intercepts is

A. $b\beta$

B. $b\beta-c$

C. $\dfrac{b\beta}{a}$

D. $\dfrac{b\beta-c}{a}$

E. $\dfrac{b\beta}{a}-c$

Question 4 Consider $f(x)=a^{\log_b x}$ where $a,b\in R^+$. The inverse of $f(x)$ is

A. $b^{\log_a x}$

B. b^{a^x}

C. a^{b^x}

D. b^{ax}

E. a^{bx}

Question 5 If the graphs of $y=f(x)$ and $y=g(x)$ intersect at (a,b), then the graphs of $y=f^{-1}(2x)$ and $y=g^{-1}(2x)$ intersect at

A. $\left(\frac{a}{2},\frac{b}{2}\right)$

B. $\left(-\frac{a}{2},-\frac{b}{2}\right)$

C. $\left(b,-\frac{a}{2}\right)$

D. $\left(-\frac{b}{2},a\right)$

E. $\left(\frac{b}{2},a\right)$

Question 6 $f(x)$ is a quadratic function.

Given $f(1)=a$, $f(3)=b$, $f'(p)=0$ and $\frac{b-a}{2}>f'(q)$, where $a>b$, $1<p<3$ and $1<q<3$.

Which one of the following statements **cannot** be true?

A. $f'(3)>f'(q)$

B. $f'(3)<f'(q)$

C. $f'(q)>0$

D. $1<q<p$

E. $p<q<3$

Question 7 Given $\int_a^b f(x)dx=c$ where $a\neq b$, the value of $\int_{\left(\frac{1-a}{2}\right)}^{\left(\frac{1-b}{2}\right)} f(1-2x)\,dx$ is

A. c

B. $\frac{c}{2}$

C. $2c$

D. $-\frac{c}{2}$

E. $-c$

Question 8 For $a, b \in R^+$, the graphs of $y = \sqrt{1 - ax}$ and $y = \sqrt{x - b}$ do **not** intersect when

A. $a + b < 1$

B. $a + b > 1$

C. $a < b$

D. $ab < 1$

E. $ab > 1$

Question 9 Line $y = x + k$ is a tangent to the curve given by $y = e^{kx} + k$ when

A. $k = 1$

B. $k = 0.74$

C. $k = 0.37$

D. $k = e^{-1}$

E. $k = e^{-2}$

Question 10 The graph of $y = ax^2(x+1)(x-1) + b$ for $a \in R^+$ does **not** cross the *x*-axis when

A. $b > \frac{a}{8}$

B. $b < \frac{a}{8}$

C. $b > \frac{a}{4}$

D. $b < \frac{a}{4}$

E. $b < \frac{a}{\sqrt{2}}$

Question 11 The graph of the **derivative** of $f(x) = ax^3 + bx^2 + cx + d$ does **not** cross the *x*-axis when

A. $c > b$

B. $3ac > b^2$

C. $4ac > b^2$

D. $b \leq -2\sqrt{c}$

E. $ad > bc$

Question 12 A solution to $\sin(2x)-\cos\left(2b-\frac{\pi}{4}\right)=0, b\in R$ is

A. $x=\frac{9\pi}{8}+b$

B. $x=\frac{11\pi}{8}-b$

C. $x=\frac{19\pi}{8}+b$

D. $x=\frac{21\pi}{8}-b$

E. $x=\frac{23\pi}{8}+b$

Question 13 $f(x)=\sin(mx)+\cos(nx)$ is a periodic function, m, n are non-zero integers.
The period of $f(x)$ is

A. π

B. 2π

C. $(m-n)\pi$

D. $(m+n)\pi$

E. $mn\pi$

Question 14 Given integers $a>b\geq 2$, the value of $\int_{a^{-1}}^{b^{-1}}\frac{b}{(bx+1)\log_e a}dx$ is

A. $\log_e\left(\frac{a}{a+b}\right)$

B. $\log_a\left(\frac{a}{a+b}\right)$

C. $\log_e\left(\frac{2a}{a+b}\right)$

D. $\log_a\left(\frac{2a}{a+b}\right)$

E. $\log_e(2a)-\log_a(a+b)$

Question 15 Two dice are rolled and the uppermost numbers are noted.
Which one of the following choices is **not** a random variable?

A. The number of even numbers

B. The sum of the numbers

C. The numbers are the same

D. The difference between the numbers

E. The number of numbers greater than 3

Question 16 $\Pr\left((A \cup B)'\right) + \Pr\left((A \cap B)'\right) =$

A. $\Pr(A') + \Pr(B')$

B. $\Pr(A) + \Pr(B)$

C. $1 - \Pr(A') - \Pr(B')$

D. $1 - \Pr(A) - \Pr(B)$

E. $\Pr(A') + \Pr(B') - 1$

Question 17 A fair coin was tossed two times and heads appeared both times.
The probability that *the third time is head and the fourth time is tail* is

A. $\frac{1}{16}$

B. $\frac{3}{16}$

C. $\frac{1}{4}$

D. $\frac{1}{3}$

E. $\frac{3}{4}$

Question 18 The probability density function of random variable X is given by

$$f(x) = \begin{cases} a\sin(2x) + \frac{\pi}{4} & \text{for } 0 \le x \le \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

The value of a is

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. $1 - \frac{\pi^2}{8}$

D. $\frac{\pi}{4} - 1$

E. $\frac{\pi}{4}$

Question 19 A probability experiment consists of 4 Bernoulli trials.
Given the probability of 2 successes is 0.0600, the probability of 3 successes is closest to

A. 0.0051

B. 0.0093

C. 0.0218

D. 0.0453

E. 0.4911

Question 20 A random sample of size 100 is taken from a large population with 80% having certain attribute.
The 80% confidence interval for the proportion of the sample with the certain attribute is closest to

A. $(0.7663,\ 0.8337)$

B. $(0.7487,\ 0.8513)$

C. $(0.7216,\ 0.8784)$

D. $(0.7211,\ 0.8789)$

E. $(0.6598,\ 0.9402)$

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise stated.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 (12 marks)
Consider $f(x) = x^4 - ax^2 + b$ where $a, b \in R$.

a. Find the restrictions on a and b such that solution(s) to $f(x) = 0$ exist. 3 marks

b. Find the values of a such that the graph of $y = x^4 - ax^2 + b$ has a local maximum on the y-axis.

2 marks

c. Let $b = 1$. Show working in finding the value of a such that the x-intercepts of the graph of $y = x^4 - ax^2 + 1$ are local minima.

2 marks

d. Given $f(x) = x^4 - 5x^2 + 4$ and $g(x) = f(x - \alpha)$, find the values of α such that the graph of $y = g(x)$ has one x-intercept with positive x-coordinate, and three x-intercepts with negative x-coordinate.

2 marks

e. The graph of $y = x^4 - 4x^2 + \beta$ has four x-intercepts.
Find the value of β such that the x-intercepts are spaced out equally.

3 marks

Question 2 (12 marks)

Consider $f(x)=\sqrt{2x}\sin(2x)$.

a. Show that the coordinates of intersections of the graphs of $y=\sqrt{2x}\sin(2x)$ and $y=\sqrt{2x}$ are $\left(\frac{\pi}{4},\sqrt{\frac{\pi}{2}}\right)$ and $\left(\frac{5\pi}{4},\frac{\sqrt{10\pi}}{2}\right)$ for $x\in(0,4]$. 2 marks

b. Show that the graphs of $y=\sqrt{2x}\sin(2x)$ and $y=\sqrt{2x}$ touch (but do not cross) each other at the intersections.

2 marks

c. The average rate of change of $f(x)=\sqrt{2x}\sin(2x)$ between the two intersections in part a with respect to x can be expressed in the form $\frac{\sqrt{p}-1}{\sqrt{q}}$. Find the value of each of p and q. 2 marks

d. Find the area of the region bounded by the graphs of $y=\sqrt{2x}\sin(2x)$ and $y=\sqrt{2x}$ for $x\in(0,4]$, correct to 4 decimal places.

1 mark

e. Show that at the local maxima of $y=\sqrt{2x}\sin(2x)$, $4x+\tan(2x)=0$. Hence find the coordinates of the local maxima for $x\in(0,4]$, correct to 4 decimal places.

2 marks

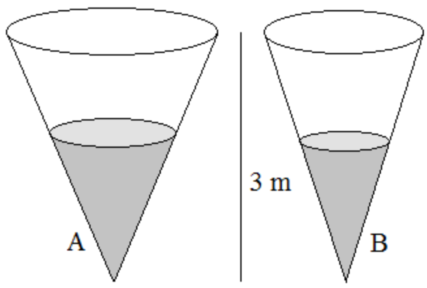
Now consider $g(x)=\sqrt{nx}\sin(nx)$ where $n \in R^+$.

f. The gradient of the line joining the first two intersections of the graphs of $y=\sqrt{nx}\sin(nx)$ and $y=\sqrt{nx}$, where $x \in \left(0, \frac{8}{n}\right]$, can be expressed in the form $k\left(\frac{\sqrt{5}-1}{2}\right)$. Find k in terms of n. 3 marks

Question 3 (12 marks)

Two cone-shape water tanks are shown in the following diagram. Initially ($t=0$), the large tank, labeled as A, is empty, and the small tank, labeled as B, is full.

Both tanks have the same height of 3 m. Volume of a cone $=\frac{\pi}{3}\times(\text{radius})^2\times\text{height}$.



The depths of water in the tanks are controlled such that both depths change at 0.1 m min^{-1}.

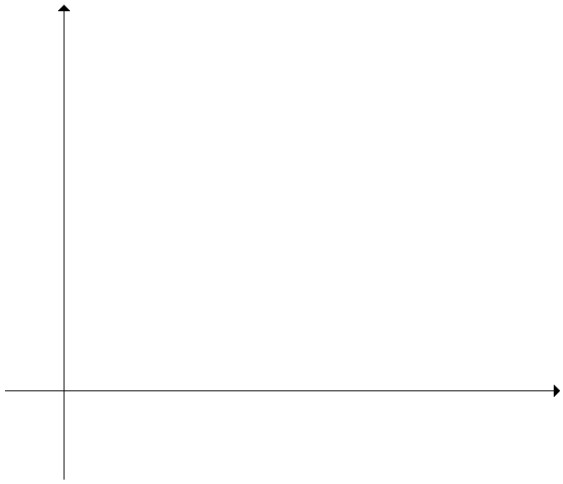
The volume (m^3) of water in Tank A and Tank B at time t (minutes) are given by $V_A=\frac{\pi t^3}{12000}$ and $V_B=\frac{\pi(30-t)^3}{27000}$ respectively.

a. Find the time to completely fill Tank A. 1 mark

b. Find the value of the ratio $\frac{r_A}{r_B}$ where r_A and r_B are the radii of Tank A and Tank B respectively. 1 mark

c. Explain that at $t=15$ both tanks have the same depth of water. 1 mark

d. Sketch the graphs of $V_{\mathrm{A}}(t)$ and $V_{\mathrm{B}}(t)$ on the same set of axes. 2 marks



e. Find the average rate of change of volume of water in each tank from $t = 0$ to $t = 15$. 3 marks

f. The two tanks have the same volume of water at $t = \dfrac{30}{1 + a^{\frac{1}{a}}}$. Find the value of a. 2 marks

g. Find the time when Tank A fills and Tank B empties at the same rate. 2 marks

Question 4 (12 marks)

In ideal conditions sound intensity level L (measured in dB) varies with distance x (metres) from a source (e.g. a siren) according to $L = 10\log_{10}\left(\frac{k}{x^2}\right)$ where $k \in R$.

a i. The sound intensity level of a siren at 30 m away is 100 dB. Show that $k = 9\times10^{12}$ m^2 for **this** siren.

1 mark

a ii. Calculate the sound intensity level of the siren at 100 m away.
Express your answer in terms of $\log_{10} 9$. 1 mark

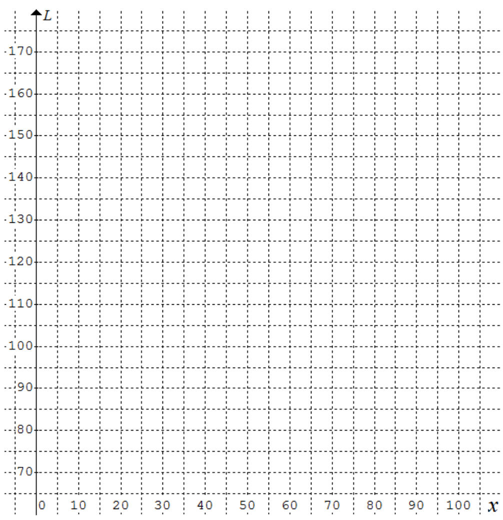
a iii. The distance from the siren is d metres when the sound intensity level is n dB.
Express d in terms of n. 1 mark

a iv. When the distance from the siren changes from x_1 to x_2, the sound intensity level drops by 6 dB.

Determine the value of the ratio $\frac{x_2}{x_1}$. 2 marks

a v. Sketch the graph of $L = 10\log_{10}\left(\frac{9\times10^{12}}{x^2}\right)$ for $x \le 100$. Show and label the distinctive features.

3 marks



a vi. The value of k is related to the power of a siren. A second more powerful siren has $k = 1.8\times10^{13}$ m^2.
Find the difference in the sound intensity levels at the same distance away from the two sirens.

1 mark

b. Find the rate of change of L with respect to x for any siren in ideal conditions at $x = \frac{10}{\log_e 20}$.

Express your answer in terms of $\log_{10}$. 3 marks

Question 5 (12 marks)
The weights (in kg) of cats in certain region have a normal distribution. 15% of the cats in a random sample of 100 from the region weigh more than 1.6 kg, and 10% weigh less than 1.25 kg.

a. A confidence interval (a, b) for the proportion of cats in the region weighing more than 1.6 kg is determined.
Given that $\Pr(0.15 \in (a, b)) = 0.95$, find the value of each of a and b, correct to 2 decimal places.

3 marks

For the following parts, **assume** that 15% of the cats in the region weigh more than 1.6 kg, and 10% weigh less than 1.25 kg.

b. Find the mean and standard deviation of weights of cats in the region, correct to 2 decimal places.

2 marks

c. Find the probability that cats in the region weighing more than 1.25 kg weigh more than 1.5 kg, correct to 2 decimal places.

2 marks

A random sample of 100 cats from the region is taken.

d. Find the probability that more than 75 cats weigh between 1.25 kg and 1.6 kg, correct to 2 decimal places.

2 marks

e. Show that the mean and standard deviation of the number of cats weighing between 1.25 kg and 1.6 kg are 75 and 4.3301 respectively.

1 mark

f. Use normal approximation to answer part d. Correct to 2 decimal places. 1 mark

g. Explain the discrepancy between your answers to part d and part f. 1 mark

End of Examination 2

