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2022 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

Application Task

Theme: Stars

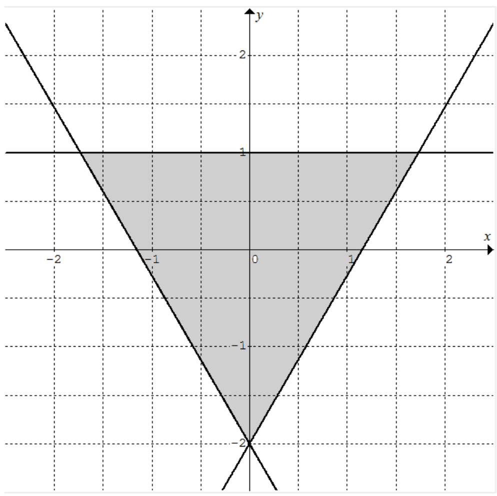
Assumed knowledge:

Functions, graphs, equations, co-ordinate geometry, transformations, 2×2 matrices, differentiation, integration, area of region bounded by curves, CAS

Information and specifications:
In this task only stars of regular shape (showing rotational symmetry) are considered.
Angles are measured in radians.
In this task a regular star must have vertices with the same angle in the interval $[0, \pi)$.
The stars can be constructed using straight lines only or curves only.
A star with n vertices requires a minimum of n straight lines or curves for its construction.
Correct answers to 3 decimal places unless stated otherwise, e.g. express answer in exact form.

Part I (80 minutes plus)

The following shape (shaded) shows a regular star with 3 vertices (an equilateral triangle).



a. The equation of the horizontal side is $y=1$.

Determine the exact equations of the other two sides using vertex angle of $\frac{\pi}{3}$, i.e. $60°$, and the same y-intercept $(0,-2)$ for both sides.

b. Determine the exact coordinates of the two vertices on the horizontal line, and find the area of the shaded region.

The other two sides are the clockwise and anticlockwise rotations of the horizontal line $y=1$ by $\frac{2\pi}{3}$, i.e. $120°$ about the origin O.

Transformation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix about the origin $O\ (0, 0)$.

Positive value of θ represents an anticlockwise rotation. For clockwise rotation θ is a negative value.

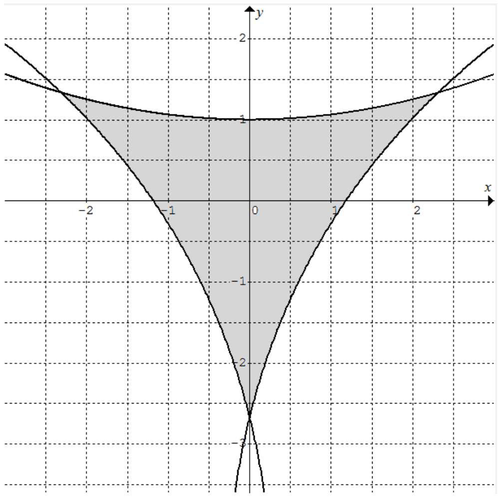
c. Write a matrix equation in finding the image point (x', y') after a rotation of the point (x, y) about the origin O by an angle of θ.

d. Apply the rotation matrix on line $y=1$ with $\theta = \pm\frac{2\pi}{3}$ to show the equations of the other two lines are $y\cos\left(\frac{2\pi}{3}\right) = 1 - x\sin\left(\frac{2\pi}{3}\right)$ and $y\cos\left(\frac{2\pi}{3}\right) = 1 + x\sin\left(\frac{2\pi}{3}\right)$.

e. Show that the equations in part a and part d are equivalent.

A regular star with 3 vertices can also be constructed by rotating about origin O a parabolic curve of the form $y=\left(\frac{x}{n}\right)^2+1$ where parameter $n>0$.

An example with $n=4$ is shown below.



f. Discuss the effects of changing the value of parameter n on the regular star.

g. Show that $n \geq 2\sqrt{3}$.

h. For $n=2\sqrt{3}$, find the angle between two curves at a vertex.

i. Describe the shape of the star as $n \to \infty$.

j. Select an n value greater than 4.
Find the equations of the three sides for your selected n value.
Express your answers in exact form with y as the subject and **without** using sine or cosine.

j. For your selected n value find the angle (correct to 3 decimal places) between two curves at a vertex.

Hint: $\tan\alpha = \frac{dy}{dx}$ where α is the angle with the x-axis made by a tangent to a curve.

k. For your chosen n value find the area of the region enclosed by the three sides, i.e. the area of the star. Show working.

End of Part I

Part II (80 minutes plus)

Reprint of Part I **Information and specifications:**
In this task only stars of regular shape (showing rotational symmetry) are considered.
Angles are measured in radians.
In this task a regular star must have vertices with the same angle in the interval $[0, \pi)$.
The stars can be constructed using straight lines only or curves only.
A star with n vertices requires a minimum of n straight lines or curves for its construction.
Correct answers to 3 decimal places unless stated otherwise, e.g. express answer in exact form.

Transformation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix about the origin $O\ (0, 0)$.

Positive value of θ represents an anticlockwise rotation. For clockwise rotation θ is a negative value.

In Part I, a regular star of 3 vertices is constructed by rotating the curve of a 'simple' **even** function of equation $y = f(x)$ about the origin O by an angle of $\theta = \pm\frac{2\pi}{3}$.

Read the following information.

For a regular star of m vertices, $\theta = \pm k\left(\frac{2\pi}{m}\right)$ where $k = 1, 2, 3, \ldots$, and the image of $y = f(x)$ after a rotation of θ is $-x\sin\theta + y\cos\theta = f(x\cos\theta + y\sin\theta)$.

$k = 1$ for a 3-vertex star, $k = 1, 2$ for a 4-vertex or 5-vertex star, $k = 1, 2, 3$ for a 6-vertex or 7-vertex star etc

Consider $y = -\left(\frac{x}{n}\right)^2 + 1$ where parameter $n > 0$.

a. When $y = -\left(\frac{x}{n}\right)^2 + 1$ is used to construct 3-vertex regular stars, show that $n > \frac{2}{\sqrt{5}}$.

b. Select an n value greater than 4.
Find the equations of the three sides for your selected n value.
Express your answers in exact form with y as the subject and **without** using sine or cosine.

c. Sketch the 3-vertex regular star for your chosen n value.

d. Find the equations, in terms of n, of the three sides of a 3-vertex regular star for any $n > \frac{2}{\sqrt{5}}$.
Express your answers in exact form with y as the subject and **without** using sine or cosine.

e. Find the exact coordinates of vertices of a 3-vertex regular star for any $n > \frac{2}{\sqrt{5}}$ in terms of n.

f. Calculate the exact value of the vertex angle of the 3-vertex regular star for $n = 4$.

g. Find a formula for ϕ_n the vertex angle of a 3-vertex regular star for any $n > \frac{2}{\sqrt{5}}$.

h. Find ϕ_∞, the limit of the vertex angle of a 3-vertex regular star as $n \to \infty$.

i. For $n = 4$ the equations of the three sides are
Calculate the exact area of the region enclosed by the 3-vertex regular star for $n = 4$.

j. Find a formula for A_n the area of the region enclosed by the 3-vertex regular star for any $n > \frac{2}{\sqrt{5}}$.

k. Find A_∞, the limit of the area of the region enclosed by the 3-vertex regular star as $n \to \infty$.

End of Part II

Part III (80 minutes plus)

Reprint of Part I and II **Information and specifications:**
In this task only stars of regular shape (showing rotational symmetry) are considered.
Angles are measured in radians.
In this task a regular star must have vertices with the same angle in the interval $[0, \pi)$.
The stars can be constructed using straight lines only or curves only.
A star with n vertices requires a minimum of n straight lines or curves for its construction.
Correct answers to 3 decimal places unless stated otherwise, e.g. express answer in exact form.

Transformation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix about the origin $O\ (0, 0)$.

Positive value of θ represents an anticlockwise rotation. For clockwise rotation θ is a negative value.

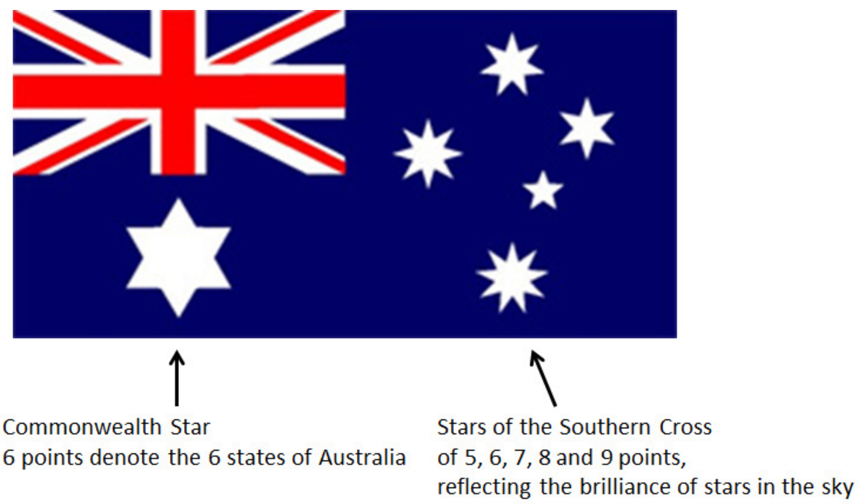
In Part I a regular star of 3 vertices is constructed by rotating the curve of a 'simple' **even** function of equation $y = f(x)$ about the origin O by an angle of $\theta = \pm\frac{2\pi}{3}$.

Read the following information.

For a regular star of m vertices, $\theta = \pm k\left(\frac{2\pi}{m}\right)$ where $k = 1, 2, 3, \ldots,$ and the image of $y = f(x)$ after a rotation of θ is $-x\sin\theta + y\cos\theta = f(x\cos\theta + y\sin\theta)$.

$k = 1$ for a 3-vertex star, $k = 1, 2$ for a 4-vertex or 5-vertex star, $k = 1, 2, 3$ for a 6-vertex or 7-vertex star etc.

The following diagram shows the original Australian National Flag (1901 – 1908).
(Image from ANFA)



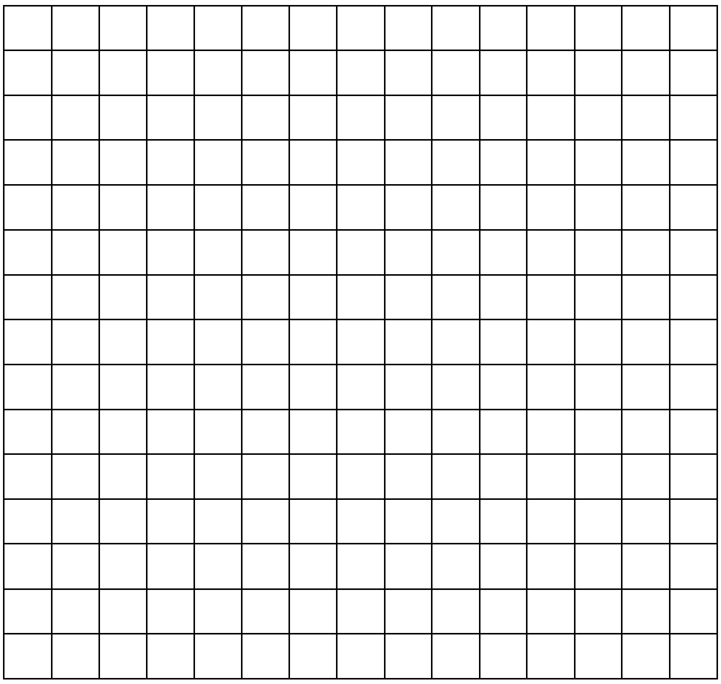
Stars of the Southern Cross on the flag are all different. The number of vertices (points) of a star ranges from 5 to 9 (increment of 1).

Your task in Part III is to construct the five stars using only straight lines by rotating horizontal line $y = 1$.
Your constructed stars may not be in the same orientation as that in the flag.

a. One of the rotation angle is $\theta = -\frac{2\pi}{5}$ for a 5-vertex regular star starting from horizontal line $y = 1$.
Write down the other rotation angles to complete the star.

b. Apply the rotation matrix on line $y = 1$ for each of the rotation angle in part a.
Find the equations of the other four lines in terms of sine and cosine.

c. Sketch the 5-vertex regular star.



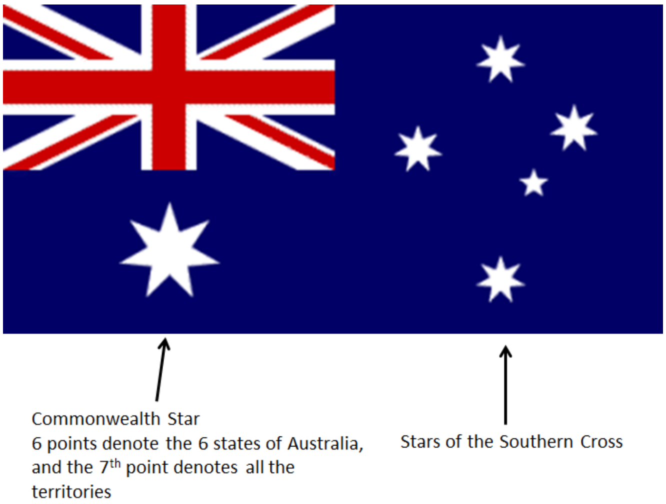
d. Determine the coordinates of the second vertex from the top vertex.

e. Calculate the vertex angle of the 5-vertex regular star discussed in parts a, b and c.

f. Calculate the area of the region enclosed by the 5-vertex regular star discussed in parts a, b and c.

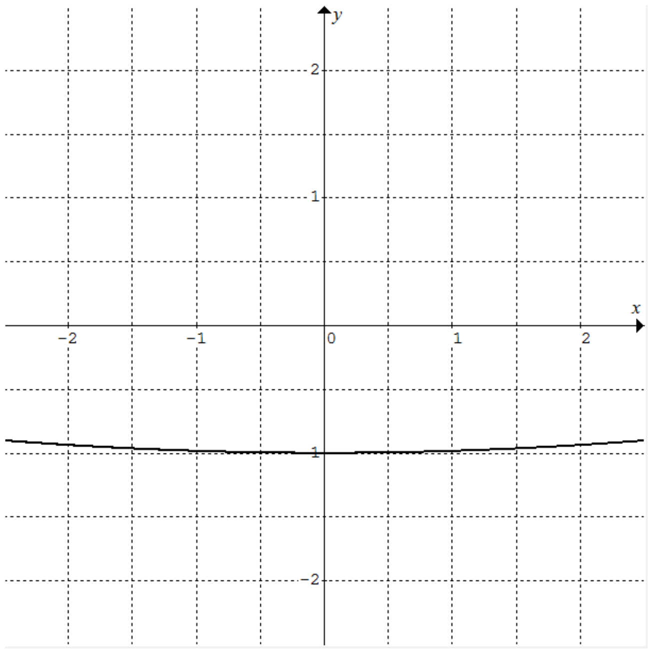
g. Write down the rotation angles to complete a regular 8-vertex star.

The following diagram shows the Australian National Flag after 1908.
(Image from ANFA)



It has a 5-vertex star and five 7-vertex stars.
Take a close look at the largest 7-vertex star on the flag.
It does not appear to be constructed with straight lines.

Assume that it is constructed using a parabolic curve $y = \left(\frac{x}{n}\right)^2 - 1$ as shown below.



h. Study the star in the flag and choose a suitable n value in $y = \left(\frac{x}{n}\right)^2 - 1$ as your starting curve for constructing the largest 7-vertex regular star shown in the flag.
Try different n values before choosing a suitable one.

i. By means of different rotation matrices determine the equations of the curves for the construction of the 7-vertex regular star.

j. Determine the coordinates of the highest and the second highest intersecting points of the curves.

k. Determine the area of the region enclosed by the 7-vertex regular star of your construction.

End of Part III

End of Application Task