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# 2022 Mathematical Methods

# Year 12

## Modelling Task

**Time allowed: 2 hours plus**

# Modelling Task

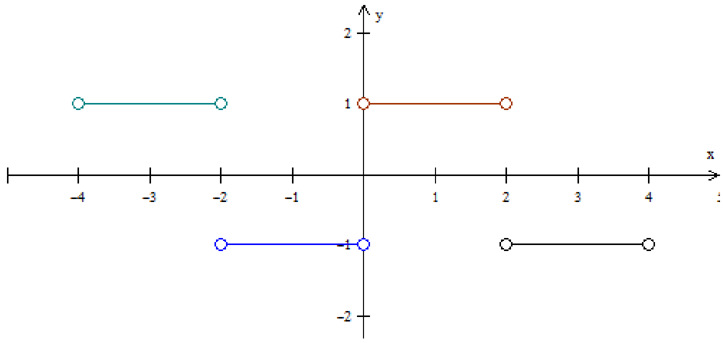
**Theme: Square waves**

**Assumed knowledge:**

Functions, graphs, inverse, transformations, equations, calculus, area of region bounded by curves and CAS

**Part I (60 minutes plus)**

The following function shows only two cycles of a periodic square wave of equation $y = f(x)$.




a. Write a piecewise function for the square wave in the interval $(-2, 2)\setminus\{0\}$.

b. Write a **general** piecewise function for the periodic square wave in terms of parameter $n \in Z$, a set of integers, given $n = 0$ for part a.

c. Sketch the cycle of the square wave for $n = -2$.

d. Write a function equation to show $f(x)$ is **periodic**.
Verify the validity of your function equation by substituting a value for $x$.

e. Write a function equation to show $f(x)$ is **odd**.
Verify the validity of your function equation by substituting a value for $x$.

f. Find the area bounded by $m$ cycles of $f(x)$ and the $x$-axis in terms of $m$ where $m \in R^{+}$.

The next task is to investigate some approximate representations of the square wave.
Firstly consider the odd function $g(x)=x^{p}$ where $p \geq 3$ and $p$ is odd.
g. For $p=3$, find $g^{-1}(x)$ and sketch its graph in the interval $(-2, 2)\setminus\{0\}$.

h. Investigate the effects of changing the value of $p$ in $g(x)=x^{p}$ on $g^{-1}(x)$.
Suggestion: Select at least 3 suitable values of $p \in (10, 150)$ and compare the corresponding graphs of $g^{-1}(x)$ in the interval $(-2, 2)\setminus\{0\}$. Comment

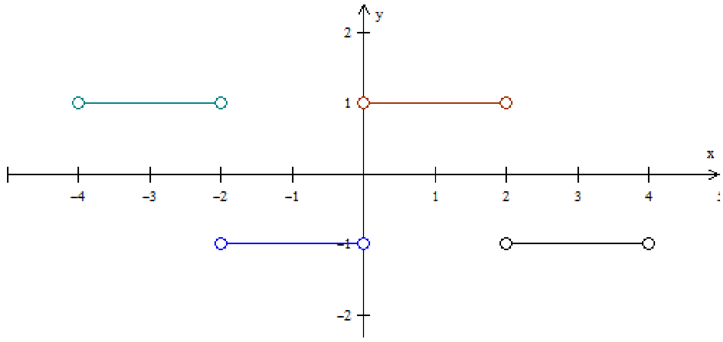
i. For **each one** of your selected $p$ values find the **difference** in areas between the regions bounded by the square wave and the *x*-axis, and the regions bounded by $g^{-1}(x)$ and the *x*-axis in the interval $(0,1)$. Comment

j. Let $\Delta A$ be the difference in areas between the regions bounded by the square wave and the *x*-axis, and the regions bounded by $g^{-1}(x)$ and the *x*-axis in the interval $(0,1)$.
For certain $p$ value $g^{-1}(x)$ is a good approximation to the square wave if $\Delta A < 0.01$ in the interval $(0,1)$.
Find the smallest value of $p$ for $g^{-1}(x)$ a good approximation.
Show working/solution process.

**End of Part I**

**Part II (60 minutes plus)**

Information from Part I:
The following function shows only two cycles of a periodic square wave of equation $y = f(x)$.




Consider the odd function $h(x) = \alpha \tan\left(\frac{\pi x}{2}\right)$ where $\alpha \in R^+$.

a. Explain why function $h(x)$ is odd.

b. Explain why function $h(x)$ is periodic.

c. For $h(x) = \alpha \tan\left(\frac{\pi x}{2}\right)$, $x \in (-1, 1) \setminus \{0\}$ and $\alpha = \frac{1}{2}$, find $x$ when $h(x) = -2$.

d. For $h(x) = \alpha \tan\left(\frac{\pi x}{2}\right)$, $x \in (-1, 1) \setminus \{0\}$ and $\alpha = \frac{1}{2}$, find $h^{-1}(x)$ and sketch its graph in the interval $(-2, 2) \setminus \{0\}$. Label all end points with coordinates.

e. Investigate the effects of changing the value of $\alpha$ in $h(x)=\alpha\tan\left(\frac{\pi x}{2}\right)$ on $h^{-1}(x)$.

Suggestion: Select at least 3 suitable values of $\alpha \in (0,1)$ and compare the corresponding graphs of $h^{-1}(x)$ in the interval $(-2,2)\setminus\{0\}$. Comment

f. For certain $\alpha$ value $h^{-1}(x)$ is a good approximation to the square wave if $h^{-1}(1)>0.995$. Find the largest value of $\alpha$ (correct to 4 decimal places) for $h^{-1}(x)$ a good approximation. Show working/solution process.

The periodic square wave can also be approximated by a series of sine functions:

$s_k(x)=\beta\left(\frac{\sin(1\pi x)}{1}+\frac{\sin(3\pi x)}{3}+\frac{\sin(5\pi x)}{5}+\cdots+\frac{\sin(k\pi x)}{k}\right)=\beta\sum_{n=1}^{k}\frac{\sin(n\pi x)}{n}$ where $\beta \in R^{+}$ and $n=1,3,5,\cdots,k$ are all odd numbers.

Let $\beta=1$ for part g and part h.

g. Investigate and comment on the resulting graph of $s(x)$ when more terms are include in the series.

Suggestion: Sketch and compare $s_5(x)=\frac{\sin(1\pi x)}{1}+\frac{\sin(3\pi x)}{3}+\frac{\sin(5\pi x)}{5}$,

$s_7(x)=\frac{\sin(1\pi x)}{1}+\frac{\sin(3\pi x)}{3}+\frac{\sin(5\pi x)}{5}+\frac{\sin(7\pi x)}{7}$ and

$s_9(x)=\frac{\sin(1\pi x)}{1}+\frac{\sin(3\pi x)}{3}+\frac{\sin(5\pi x)}{5}+\frac{\sin(7\pi x)}{7}+\frac{\sin(9\pi x)}{9}$ in the interval $(-2,2)\setminus\{0\}$.

h. When $k \to \infty$, estimate (correct to 2 decimal places) the **average** value of $s_k(x)$ in the interval $(0, 1)$.

i. Hence determine the value of $\beta$ (correct to 1 decimal place) such that $s_k(x)$ approximates the periodic square wave as $k \to \infty$.

j. Discuss one significant advantage of using

$$s_k(x) = \beta\left(\frac{\sin(1\pi x)}{1} + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \cdots + \frac{\sin(k\pi x)}{k}\right)$$

over the inverse of $g(x) = x^p$ and the inverse of $h(x) = \alpha \tan\left(\frac{\pi x}{2}\right)$ to approximate the periodic square wave.

**End of Part II**