2022 NSW ESA Mathematics Advanced Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	D	B	A	D	B	C	C	A	B

Q1 Negative gradient, positive y-intercept **A**

Q2 **D**

Q3 $\frac{h}{AB} = \tan 26°$, $AB = \frac{h}{\tan 26°} = h\cot 26°$ **B**

Q4 Minimum is at $(0, -1)$. Range is $[-1, \infty)$ **A**

Q5 $h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \frac{8\times 4 - 2\times 12}{8^2} = \frac{1}{8}$ when $x = 1$ **D**

Q6 $\int \frac{1}{(2x+1)^2}dx = \int (2x+1)^{-2}dx = \frac{(2x+1)^{-1}}{-1\times 2} + c = \frac{-1}{2(2x+1)} + c$ **B**

Q7 The mode is the x-coordinate of the maximum point. **C**

Q8 $\int_{-2}^{2} f(x)dx = -B + A + A - B = -2$ **C**

Q9 $\Pr(X \geq 1) = \Pr(WW') + \Pr(W'W) + \Pr(WW) = 0.80$

$2p(1-p) + p^2 = 0.80$, $2p - p^2 = 0.80$, $p^2 - 2p + 0.80 = 0$

$p = \frac{2 - \sqrt{4 - 3.2}}{2} \approx 0.5528$, $p^2 \approx 0.31 = 31\%$ **A**

Q10 $y = f(x)$ is an even function $\therefore f(-a) = f(a) = b$

$\therefore g(f(\pm a)) = g(b) = c$ is also an even function. **B**

Section II

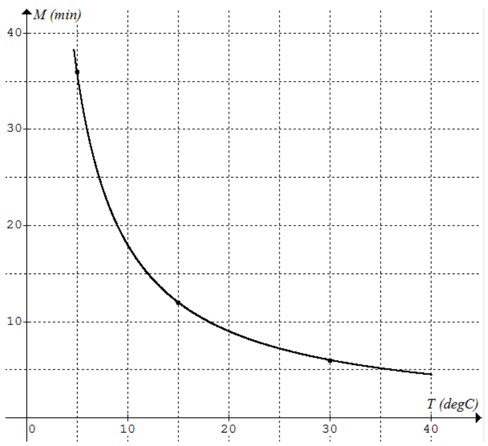
Q11 $A = 98 + 62 = 160$, $B = \frac{192}{200}\times 100 = 96$

Q11b The first two on the Pareto chart, i.e. Stock shortage and Delivery fee.

Q12a $M = \frac{k}{T}$ and $(15, 12)$, $12 = \frac{k}{15}$, $k = 180$, $M = \frac{180}{T}$

Q12b

T	5	15	30
M	36	12	6



Q13 $f(x) = \sqrt{1+x^2}$, $f(0) = 1$, $f(1) = \sqrt{2}$, $f(2) = \sqrt{5}$

$\int_0^2 \sqrt{1+x^2}\,dx \approx \frac{1+\sqrt{2}}{2} + \frac{\sqrt{2}+\sqrt{5}}{2} \approx 3.03$

Q14 Amplitude $k = 4$, period $T = \frac{2\pi}{a} = 6\pi$ $\therefore a = \frac{1}{3}$

Q15a $\Pr(5) = \frac{1}{3}\times\frac{1}{6} + \frac{1}{3}\times\frac{1}{6} + \frac{1}{3}\times\frac{1}{2} = \frac{5}{18}$

Q15b $\Pr(\text{special die} \mid 5) = \frac{\Pr(\text{special die} \cap 5)}{\Pr(5)} = \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{5}{18}} = \frac{3}{5}$

OR Total number of 5s $= 5$, special die has three 5s,

$\Pr(\text{special die} \mid 5) = \frac{3}{5}$

Q16 $\int_{-1}^{3} 2x + 3 - x^2\ dx = \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3} = 9 - \left(-2 + \frac{1}{3}\right) = \frac{32}{3}$

Q17a Arith. Sequence: $a = 3$, $d = 3$,

$S_{12} = \frac{12}{2}(2\times 3 + 11\times 3) = 234$

Q17b $\frac{n}{2}(2\times 3 + (n-1)\times 3) = 828$, $\frac{3n(n+1)}{2} = 828$,

$n(n+1) = 552 = 23\times 24$ $\therefore n = 23$

Q18a By the chain rule: $\frac{dy}{dx} = 4(x^2+1)^3(2x) = 8x(x^2+1)^3$

Q18b From part a, $x(x^2+1)^3 = \frac{1}{8}\frac{dy}{dx}$,

$\int x(x^2+1)^3 dx = \frac{1}{8}\int \frac{dy}{dx}dx = \frac{1}{8}\int dy = \frac{1}{8}(x^2+1)^4 + c$

Q19 $kf(x-m) - 5 = g(x)$, $k(x-m)^2 - 5 = 3x^2 - 12x + 7$, $m > 0$

$\therefore k = 3$ and $km^2 - 5 = 7$ $\therefore m = 2$

Q20a $N(0) = 200$

Q20b $N(24) = 200e^{0.013\times 24} \approx 273$ (273.23)

Q20c $\frac{dN}{dt} = 0.013\times 200e^{0.013t}$, at $t = 24$,

$\frac{dN}{dt} = 0.013\times 200e^{0.013\times 24} = 0.013\times 273.23 \approx 3.55$

Q21a

$A = 40000\left(1 + \frac{1.2\%}{12}\right)^{10\times 12} = 40000\times 1.001^{120} \approx 45097.17$ dollars

Q21b $Rate = \frac{2.4\%}{4} = 0.006$, $N = 10\times 4 = 40$

Use table: $A = 45.05630\times 1000 = 45056.30$

Difference $\approx 45056.30 - 45097.17 \approx -40.87$

Future value using Option 2 is $40.87 less than using Option 1.

Q22 $y = f(x) = x^3 - 6x^2 + 8$, $-1 \le x \le 7$, $f(-1) = 1$, $f(7) = 57$

Let $\frac{dy}{dx} = 3x^2 - 12x = 0$, $x = 0, 4$

.: local max point is $(0, 8)$ and local min point is $(4, -24)$

.: global min value of $y = f(x)$ is -24 and global max value is 57.

Q23a $d_{low} = 1.3 - 0.6 = 0.7$ m, $d_{high} = 1.3 + 0.6 = 1.9$ m

Q23b Period $T = \frac{2\pi}{\frac{4\pi}{25}} = 12.5$ h

Q23c $d = 1.3 - 0.6\cos\frac{4\pi}{25}t \ge 1$.: $\cos\frac{4\pi}{25}t \le \frac{1}{2}$

Consider $\cos\frac{4\pi}{25}t = \frac{1}{2}$, $\frac{4\pi}{25}t = \frac{\pi}{3}, \frac{5\pi}{3}$, $t = \frac{25}{12}, \frac{125}{12}$

.: for $d = 1.3 - 0.6\cos\frac{4\pi}{25}t \ge 1$, $\frac{25}{12} \le t \le \frac{125}{12}$

.: $\Delta t = \frac{125}{12} - \frac{25}{12} = \frac{25}{3} = 8$ hours 20 min

Q24 The scatterplot shows that in general as the age of character is higher, the age of actor is also higher.
The line of best fit indicates that there is a linear relationship between the age of character and the age of actor.
The equation $y = -7.51 + 1.85x$ shows that on average for each year of increase in the age of character, there is an increase of 1.85 years in the age of actor.
Coefficient 0.4564 indicates that there is a moderate correlation between the age of character and the age of actor.

Q25 $f(x) = \sin 2x$, $f'(x) = 2\cos 2x = -\sqrt{3}$ AND $f''(x) = -4\sin 2x = 2$

.: $\cos 2x = -\frac{\sqrt{3}}{2}$ AND $\sin 2x = -\frac{1}{2}$, $0 < x < \pi$

For $\cos 2x = -\frac{\sqrt{3}}{2}$, $x = \frac{5\pi}{12}, \frac{7\pi}{12}$. For $\sin 2x = -\frac{1}{2}$, $x = \frac{7\pi}{12}, \frac{11\pi}{12}$

Only $x = \frac{7\pi}{12}$ satisfies both equations.

Q26 $\Pr(840 < X < 860) = 0.6 - 0.5 = 0.1$.: $\Pr(820 < X < 840) = 0.1$
920 is one standard deviation from the mean,

.: $\Pr(840 < X < 920) = \frac{0.68}{2} = 0.34$

.: $\Pr(820 < X < 920) = 0.1 + 0.34 = 0.44$, i.e. 44%

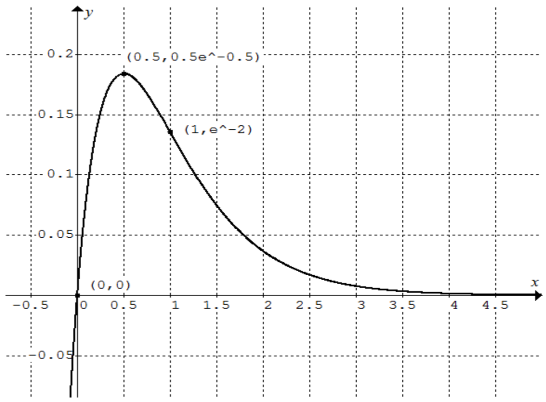
Q27a $f(x) = xe^{-2x}$, $f'(x) = e^{-2x} - 2xe^{-2x} = (1 - 2x)e^{-2x}$

$f''(x) = -2e^{-2x} - 2(1 - 2x)e^{-2x} = 4(x - 1)e^{-2x}$

Q27b $f'(x) = (1 - 2x)e^{-2x} = 0$, $x = \frac{1}{2}$, $y = f\left(\frac{1}{2}\right) = \frac{1}{2e}$.: $\left(\frac{1}{2}, \frac{1}{2e}\right)$

$f''\left(\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)e^{-1} < 0$.: $\left(\frac{1}{2}, \frac{1}{2e}\right)$ is a local maximum.

Q27c



Q28a $\theta = 45°$, shaded area $= \frac{45}{360} \times \pi \times 2 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$

Q28b $y = \frac{a}{b - x} - 1$, $(0, 0)$, $(1, 1)$.: $0 = \frac{a}{b} - 1$, $a = b$

and $1 = \frac{a}{b - 1} - 1$.: $1 = \frac{b}{b - 1} - 1$, $b = 2$.: $a = b = 2$

Q28c Line through $(0, 0)$ and $(1, 1)$ is $y = x$

Area between line $y = x$ and hyperbola $= \int_0^1 x - \left(\frac{2}{2 - x} - 1\right) dx$

$= \int_0^1 x + 1 - \frac{2}{2 - x}\, dx = \left[\frac{x^2}{2} + x + 2\ln(2 - x)\right]_0^1 = \frac{3}{2} - 2\ln 2$

Required area $= \frac{1}{8}\pi 2 - \left(\frac{3}{2} - 2\ln 2\right) = \frac{\pi}{4} - \frac{3}{2} + 2\ln 2$

Q29a Geometric sequence: $1, \frac{1}{2}, \frac{1}{4}, \cdots$ $a = 1$ and $r = \frac{1}{2}$

$S_\infty = \frac{1}{1 - \frac{1}{2}} = 2$

Q29b $\int_0^4 2^{-x} dx = \int_0^4 e^{-x\ln 2} dx = \left[\frac{e^{-x\ln 2}}{-\ln 2}\right]_0^4 = \left[\frac{2^{-x}}{-\ln 2}\right]_0^4$

$= \frac{2^{-4}}{-\ln 2} - \frac{1}{-\ln 2} = \frac{15}{16\ln 2}$

Q29c $\frac{15}{16\ln 2} < 2$, $\frac{1}{\ln 2} < \frac{32}{15}$, $15 < 32\ln 2$, $15 < \ln 2^{32}$.: $e^{15} < 2^{32}$

Q30a Continuous .: at $x = e^3$, $\frac{1}{k}\ln e^3 = 1$.: $\frac{3}{k} = 1$, $k = 3$

Q30b $P(X < c) = 2P(X > c)$, $\frac{1}{k}\ln c = 2\left(1 - \frac{1}{k}\ln c\right)$, $\frac{3}{k}\ln c = 2$,

$\frac{3}{3}\ln c = 2$.: $c = e^2$

Q31a Line through $(0, y)$, $(1, 2)$ and $(x, 0)$

.: $\frac{y - 2}{0 - 1} = \frac{2 - 0}{1 - x}$.: $y = \frac{-2}{1 - x} + 2$, $y = \frac{2x}{x - 1}$

Q31b $A=\frac{1}{2}xy=\frac{1}{2}x\times\frac{2x}{x-1}=\frac{x^2}{x-1}$ and $x>1$

$\frac{dA}{dx}=\frac{x^2-2x}{(x-1)^2}$. Let $\frac{x^2-2x}{(x-1)^2}=0$, $x^2-2x=0$.: $x=2$ since $x>1$

$\frac{d^2A}{dx^2}=\frac{(x-1)^2(2x-2)-(x^2-2x)2(x-1)}{(x-1)^3}=2>0$ when $x=2$

.: $A(2)=4$ is the minimum area

Q32a Let $n=180$, $P=200000$ and $A_{180}=0$

$M=\frac{200000(1.0025)^{180}}{1+1.0025+1.0025^2+\cdots+1.0025^{179}}$

$=\frac{200000(1.0025)^{180}}{\frac{1.0025^{180}-1}{1.0025-1}}\approx 1381.16$

Q32b Let $A_n=0$

$100032(1.0035)^n-1381.16\left(1+1.0035+1.0035^2+\cdots+1.0035^{n-1}\right)=0$

$100032(1.0035)^n-1381.16\left(\frac{1.0035^n-1}{1.0035-1}\right)=0$

$100032(1.0035)^n-394617.14\left(1.0035^n-1\right)=0$

$294585.14\times 1.0035^n=394617.14$

$1.0035^n=1.339569$, $n=\frac{\ln 1.339569}{\ln 1.0035}\approx 83.6741$

.: 83 months of full monthly repayments.

Q32c $A_{83}=100032(1.0035)^{83}-1381.16\left(\frac{1.0035^{83}-1}{0.0035}\right)\approx 928.29$

Final payment $\approx 928.29\times 1.0035\approx 931.54$ dollars

Please inform mathline@itute.com re conceptual and/or mathematical errors