2022 NSW ESA Mathematics Extension 1 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	A	D	A	B	B	C	D	D	B

Q1 $\cos\left(\frac{23\pi}{12}\right) = \cos\left(-\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{12}\right)$, $\cos^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) = \frac{\pi}{12}$ **C**

Q3 $\frac{P(x)}{Q(x)} = S(x) + \frac{2x+5}{Q(x)}$, $2x+5$ is the remainder .: $Q(x)$ must have degree 2 or higher. **D**

Q4 Check the correctness of the x-intercepts of $y = f(x) + g(x)$ by addition of ordinates of $y = f(x)$ and $y = g(x)$. **A**

Q5 $y = 3 - 2(x-2)^2$ **B**

Q6 The diagram is drawn to scale, the projection $\tilde{u}$ onto $-\tilde{i} + \tilde{j}$ is $-0.4\tilde{i} + 0.4\tilde{j}$. **B**

Q7 ${}^{12}C_3 - {}^{3}C_3 - {}^{4}C_3 - {}^{5}C_3 = 205$ **C**

Q8 $(\tilde{a}+\tilde{b}).(\tilde{a}+\tilde{b}) = |\tilde{a}+\tilde{b}|^2$ Since $|\tilde{a}+\tilde{b}| < 1$.: $\tilde{a}.\tilde{a} + \tilde{b}.\tilde{b} + 2\tilde{a}.\tilde{b} < 1$

$\tilde{a}$ and $\tilde{b}$ are unit vectors .: $1 + 1 + 2\cos\theta < 1$, $\cos\theta < -\frac{1}{2}$ **D**

Q9 The graphs of $y = \frac{1}{x}$ and its inverse show A, B and C are false. **D**

Q10 $-1 \le \sin y \le 1$.: $0 \le \sin y + 1 \le 2$ for $y \in R$

By considering the inverse of periodic $\frac{dy}{dx} = \sin y + 1$, $\sin y + 1 \ne 0$,

.: $0 < \frac{dy}{dx} \le 2$ **B**

Section II

Q11ai $\tilde{u} + 3\tilde{v} = 7\tilde{i} + 2\tilde{j}$

Q11aii $\tilde{u}.\tilde{v} = 2 - 1 = 1$

Q11b Let $u = x^2 + 4$. $\int_0^1 \frac{x}{\sqrt{x^2+4}}dx = \int_4^5 \frac{1}{2\sqrt{u}}du = \left[\sqrt{u}\right]_4^5 = \sqrt{5} - 2$

Q11c $\left(1 - \frac{x}{2}\right)^8 = 1 - {}^8C_1\frac{x}{2} + {}^8C_2\left(\frac{x}{2}\right)^2 - {}^8C_3\left(\frac{x}{2}\right)^3 + \cdots$

Coefficient of x^2 is $\frac{{}^8C_2}{4} = 7$, coefficient of x^3 is $-\frac{{}^8C_3}{8} = -7$

Q11d $u.v = a(a-7) + 2(4a-1) = 0$, $a^2 + a - 2 = 0$, $a = -2$ or 1

Q11e $R\sin(x+\alpha) = R\cos\alpha\sin x + R\sin\alpha\cos x$

$R\cos\alpha = \sqrt{3}$, $R\sin\alpha = -3$

Squaring both equations and adding: $R^2 = 12$, $R = \pm 2\sqrt{3}$

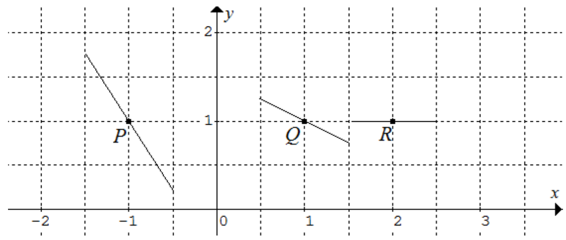
Dividing: $\tan\alpha = -\sqrt{3}$, $\alpha = -\frac{\pi}{3}, \frac{2\pi}{3}, \cdots$. Two possible forms are

$\sqrt{3}\sin x - 3\cos x = 2\sqrt{3}\sin\left(x - \frac{\pi}{3}\right)$ or $-2\sqrt{3}\sin\left(x + \frac{2\pi}{3}\right)$

Q11f If $x > 2$, $\frac{x}{2-x} < 0$.: $\frac{x}{2-x} \ge 5$ has no solutions when $x > 2$.

If $x < 2$, $x \ge 5(2-x)$, $x \ge \frac{5}{3}$.: $\frac{5}{3} \le x < 2$.

Q12a At $P(-1,1)$, $Q(1,1)$ and $R(2,1)$, $\frac{dy}{dx} = -\frac{3}{2}, -\frac{1}{2}$, and 0 respectively.



Q12b To have the least number of players above the age limit, the 41 players are spread equally between the 13 teams at 3 players each. There will be 2 players remaining to be placed in one or two of the teams. .: at least one team will be penalised.

Q12c $y = x\tan^{-1}(x)$, $\frac{dy}{dx} = \tan^{-1}(x) + \frac{x}{1+x^2}$

At $\left(1, \frac{\pi}{4}\right)$, $m = \frac{dy}{dx} = \frac{\pi}{4} + \frac{1}{2} = \frac{\pi+2}{4}$.: tangent: $y = \left(\frac{\pi+2}{4}\right)x - \frac{1}{2}$

Q12di Newton's Law of cooling $\frac{dT}{dt} = k(T - T_1)$ where $T_1 = 12$.

$\int \frac{1}{T-12}dT = k\int dt$, $\ln(T-12) = kt + c$, $(0, 92)$.: $c = \ln 80$

.: $\ln\left(\frac{T-12}{80}\right) = kt$, $(5, 76)$.: $k = \frac{1}{5}\ln\left(\frac{4}{5}\right)$.: $T = 12 + 80e^{\frac{t}{5}\ln\left(\frac{4}{5}\right)}$

Q12dii $\frac{t}{5}\ln\left(\frac{4}{5}\right) = \ln\left(\frac{T-12}{80}\right)$, when $T = 57$, $\frac{t}{5}\ln\left(\frac{4}{5}\right) = \ln\left(\frac{45}{80}\right)$

.: $t \approx 13$

Q12e For each draw the expected score is $\frac{3}{10}\times 10 + \frac{7}{10}\times {}^{-}5 = -\frac{1}{2}$.

For four independent draws, the expected score is $-\frac{1}{2}\times 4 = -2$

Q12f For $n = 0$, $15^0 + 6^1 = 7$ is divisible by 7.

Assume that for $n = k$, $15^k + 6^{2k+1}$ is divisible by 7,

i.e. $15^k + 6^{2k+1} = 7m$ where m is a positive integer.

Consider $n = k+1$. $15^{k+1} + 6^{2(k+1)+1} = 15\times 15^k + 36\times 6^{2k+1}$

$= 15\times 15^k + 15 + \times 6^{2k+1} + 21\times 6^{2k+1} = 15\left(15^k + 6^{2k+1}\right) + 7\times 3\left(6^{2k+1}\right)$

$= 15(7m) + 7\times 3\left(6^{2k+1}\right) = 7\left(15m + 3\left(6^{2k+1}\right)\right)$

.: $15^{k+1} + 6^{2(k+1)+1}$ is divisible by 7

.: $15^n + 6^{2n+1}$ is divisible by 7 for all integers $n \ge 0$.

Q13a $\overrightarrow{CA} = \tilde{a} - \tilde{c}$ and $\overrightarrow{BH} = \tilde{h} - \tilde{b} = \tilde{a} + \tilde{c}$

$\overrightarrow{BH}.\overrightarrow{CA} = (\tilde{a} + \tilde{c}).(\tilde{a} - \tilde{c}) = \tilde{a}.\tilde{a} - \tilde{c}.\tilde{c} = a^2 - c^2 = 0$, since $a = c =$ radius .: $\overrightarrow{BH} \perp \overrightarrow{CA}$

Q13b $V=\pi^2=\int_0^{\frac{\pi}{2k}}\pi(k+1)^2\sin^2(kx)\,dx=\pi(k+1)^2\int_0^{\frac{\pi}{2k}}\frac{1-\cos 2kx}{2}\,dx$

$=\frac{\pi(k+1)^2}{2}\left[x-\frac{\sin 2kx}{2k}\right]_0^{\frac{\pi}{2k}}=\frac{\pi(k+1)^2}{2}\left[\frac{\pi}{2k}\right]=\frac{\pi^2(k+1)^2}{4k}$

.: $\frac{(k+1)^2}{4k}=1$.: $k^2-2k+1=0$, $k=1$

Q13c $f(x)$ is a many to one function, .: its inverse is not a function. $g(x)=\sin^{-1}x$ is a function .: it cannot be the inverse of f.

Q13d $P(x)=x^3+bx^2+cx+d$, $b=-(\alpha+\beta+\gamma)$, $c=\alpha\beta+\beta\gamma+\gamma\alpha$

$P'(x)=3x^2+2bx+c$

$P'(\alpha)+P'(\beta)+P'(\gamma)=3(\alpha^2+\beta^2+\gamma^2)+2b(\alpha+\beta+\gamma)+3c$

$=3(\alpha^2+\beta^2+\gamma^2)+2b(-b)+3(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=3(\alpha^2+\beta^2+\gamma^2)-2(\alpha+\beta+\gamma)^2+3(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=(\alpha^2+\beta^2+\gamma^2)-(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=85-(\alpha\beta+\beta\gamma+\gamma\alpha)=87$, .: $\alpha\beta+\beta\gamma+\gamma\alpha=-2$

Q13ei Binomial: $n=16$, $p=0.2$ for less than 150 g
Normal approximation:
$\mu=np=16\times0.2=3.2$, $\sigma=\sqrt{np(1-p)}=1.6$

$\Pr(X\ge 8)=\Pr\left(Z\ge\frac{8-3.2}{1.6}\right)=\Pr(Z\ge 3)=1-\Pr(Z<3)$

$\approx 1-0.9987=0.0013$

Q13eii The sample size is too small and the distribution is too skewed for the approximation to be accurate and may not be valid.
Rule of thumb: Both $np\ge 10$ and $n(1-p)\ge 10$ for valid application of the approximation. In this case $np<10$.

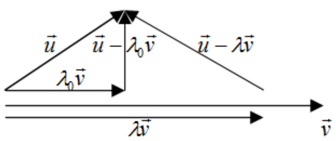
Q14a $\int\frac{1}{y}dy=\int\frac{x}{x-2}dx$, $\int\frac{1}{y}dy=\int\left(1+\frac{2}{x-2}\right)dx$

$\ln y+c=x+2\ln|x-2|=x+\ln(x-2)^2$, $(0,1)$.: $c=2\ln 2=\ln 2^2$

.: $\ln y=x+\ln\left(\frac{x-2}{2}\right)^2$, $\ln y=\ln e^x+\ln\left(\frac{x-2}{2}\right)^2$

.: $y=e^x\left(\frac{x-2}{2}\right)^2$

Q14b Since $\lambda_0\vec{v}$ is the projection of $\vec{u}$ onto $\vec{v}$.: $\vec{u}-\lambda_0\vec{v}$ is the vector resolute of $\vec{u}$ perpendicular to $\vec{v}$. For $\lambda\ne\lambda_0$, $\vec{u}-\lambda\vec{v}$ is not perpendicular to $\vec{v}$.
Perpendicular distance is the smallest,
.: $|\vec{u}-\lambda_0\vec{v}|\le|\vec{u}-\lambda\vec{v}|$ for all $\lambda\in R$.



Q14c Landing time of projectile: $2ut\sin\theta-\frac{g}{2}t^2=0$, $t>0$

i.e. $t=\frac{4u\sin\theta}{g}$

.: landing position of projectile is

$x_P=2ut\cos\theta=\frac{4u^2 2\sin\theta\cos\theta}{g}=\frac{4u^2\sin 2\theta}{g}$

and the target position is $x_T=d+ut=d+\frac{4u^2\sin\theta}{g}$

.: the target can possibly be hit when $x_T\le x_P$

.: $d+\frac{4u^2\sin\theta}{g}\le\frac{4u^2\sin 2\theta}{g}$.: $d\le\frac{4u^2\sin 2\theta}{g}-\frac{4u^2\sin\theta}{g}$

$d\le\frac{4u^2}{g}(\sin 2\theta-\sin\theta)$

Note: When $\theta=\frac{\pi}{4}$, max $x_P=\frac{4u^2}{g}$

Maximise $\sin 2\theta-\sin\theta$ to find the upper limit of d:

Let $\frac{d}{d\theta}(\sin 2\theta-\sin\theta)=0$, $2\cos 2\theta-\cos\theta=0$,

$4\cos^2\theta-\cos\theta-2=0$, $\cos\theta\approx 0.8431$, $\theta\approx 0.5678$

Maximum $\sin 2\theta-\sin\theta\approx 0.37=37\%$

.: the upper limit of d is 37% (2 sig. fig.) of the maximum possible range of the projectile.

Q14d Use the normal approximation to the binomial:
Let n be the number of tickets sold and $p=0.95$ the probability of not missing flight, and random variable X the number of passengers turning up.

.: $\mu=np=0.95n$ and $\sigma=\sqrt{0.95\times0.05n}\approx 0.217945\sqrt{n}$

Require $\Pr(X>350)\le 0.01$, $\Pr\left(Z>\frac{350-0.95n}{0.217945\sqrt{n}}\right)\le 0.01$

$\Pr\left(Z\le\frac{350-0.95n}{0.217945\sqrt{n}}\right)\ge 0.99$.: $\frac{350-0.95n}{0.217945\sqrt{n}}\ge 2.33$

$350-0.95n\ge 0.5078\sqrt{n}$, $0.95n+0.5078\sqrt{n}-350\le 0$

.: $\sqrt{n}\le 18.93$.: $n\le 358.3$.: maximum $n=358$

Please inform mathline@itute.com re conceptual and/or mathematical errors.