2022 NSW ESA Mathematics Extension 2 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	D	B	C	C	A	D	B	D	C

Q1 **A**

Q2 $a+2=\pm 2$ **D**

Q3 **B**

Q4 $\frac{x-1}{x^3-x^2+x-1}=\frac{x-1}{(x-1)(x^2+1)}=\frac{1}{x^2+1}$ **C**

Q5 $\frac{d}{dx}\left(\frac{1}{2}[g(x)]^2\right)=g(x)\frac{d}{dx}g(x)=g(x)f(x)$ **C**

Q6 Consider $z=a-ai$ for $a\neq 0$, $\bar{z}=iz$ **A**

Q7 P is false when $n=3$, the converse of P is true when $n=2$. **D**

Q8 Let $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ makes θ with the vertical, where $\cos\theta=\frac{\dot{y}}{v}$

The vertical component: $m\ddot{y}=kv^2\cos(v-\pi)-mg$

.: $m\ddot{y}=-kv^2\cos\theta-mg=-kv\dot{y}-mg$ **B**

Q9 S_1 is a spherical surface centred at midpoint M. Let its radius be r. S_2 is a plane perpendicular to and bisects AM. Circle S has its centre O at the midpoint of AM, and it is on the plane and the spherical surface.

Radius of $S=\sqrt{r^2-\left(\frac{r}{2}\right)^2}=\frac{\sqrt{3}}{2}r=\frac{\sqrt{3}\left|\overrightarrow{AB}\right|}{4}$ **D**

Q10 If the particle is initially moving downwards with non-zero speed, then its speed will eventually approach a terminal speed.
.: A and B are not always true.
If the particle is initially moving upwards, then it comes to a stop and moves downwards.
Its speed will eventually approach a terminal speed.
.: D is not true. **C**

Section II

Q11a $\frac{3-i}{2+i}\times\frac{2-i}{2-i}=1-i$

Q11b Let $u=\sin 2x$. $\int\sin^3 2x\cos 2x\,dx=\int\frac{1}{2}u^3\,du=\frac{1}{8}\sin^4 2x+c$

Q11ci $-\sqrt{3}+i=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=2e^{i\frac{5\pi}{6}}$

Q11cii $\left(-\sqrt{3}+i\right)^{10}=2^{10}\left(\cos\left(10\times\frac{5\pi}{6}\right)+i\sin\left(10\times\frac{5\pi}{6}\right)\right)$

$=2^{10}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2^{10}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=512+512\sqrt{3}\,i$

Q11d $\overrightarrow{BA}=\vec{i}-3\vec{j}+3\vec{k}$, $\overrightarrow{BC}=2\vec{i}-\vec{j}+2\vec{k}$

$\cos\angle ABC=\frac{\overrightarrow{BA}.\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|}=\frac{11}{\sqrt{19}\sqrt{9}}$. $\angle ABC\approx 33°$

Q11e ℓ_2: $\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-6\\ 5\end{pmatrix}+\mu\begin{pmatrix}3\\ 2\end{pmatrix}$; $x=-6+3\mu$ and $y=5+2\mu$

Eliminate μ to obtain $y=\frac{2}{3}x+9$

Q11f $t=\tan\frac{x}{2}$, $\cos x=\frac{1-t^2}{1+t^2}$, $\sin x=\frac{2t}{1+t^2}$,

$\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1+t^2}{2}$, $dx=\frac{2}{1+t^2}dt$

$\int\frac{1}{1+\cos x-\sin x}dx=\int\frac{1}{1-t}dt=-\ln|1-t|+c=-\ln\left|1-\tan\frac{x}{2}\right|+c$

Q12a $(a-b)^2\geq 0$, $(a-b)^2+4ab\geq 4ab$, $(a+b)^2\geq 4ab$

For $a,b\geq 0$, $a+b\geq 2\sqrt{ab}$, $\frac{a+b}{2}\geq\sqrt{ab}$

Q12b $t=0$, $x=0$, $v=0$, $\frac{dv}{dt}=12-6t$

$\frac{dx}{dt}=\int_0^t(12-6t)dt=12t-3t^2$, $x=\int_0^t\left(12t-3t^2\right)dt=6t^2-t^3$

Max velocity when $a=0$, $t=2$, $x(2)=16$

Q12ci $t=0$, $x=0$, $v=u$ to the right, $v\neq 0$

$a=\frac{F}{1}=-\left(v+3v^2\right)$, $v\frac{dv}{dx}=-\left(v+3v^2\right)$, $\frac{dv}{dx}=-(1+3v)$

Q12cii $\frac{dx}{dv}=-\frac{1}{1+3v}$,

$x=\int_u^v-\frac{1}{1+3v}dv=\left[-\frac{1}{3}\ln(1+3v)\right]_u^v=\frac{1}{3}\ln\frac{1+3u}{1+3v}$ where $u,v>0$.

Q12d $\frac{4+x}{(1-x)(4+x^2)}=\frac{A}{1-x}+\frac{Bx+C}{4+x^2}$, $A=1$, $B=1$ and $C=0$

$\int_2^n\frac{4+x}{(1-x)(4+x^2)}dx=\int_2^n\left(\frac{1}{1-x}+\frac{x}{4+x^2}\right)dx$

$=\left[-\ln|1-x|+\frac{1}{2}\ln\left(4+x^2\right)\right]_2^n=\left[-\ln\sqrt{(1-x)^2}+\frac{1}{2}\ln\left(4+x^2\right)\right]_2^n$

$=\left[\frac{1}{2}\ln\frac{4+x^2}{(1-x)^2}\right]_2^n=\frac{1}{2}\ln\frac{4+n^2}{8(1-n)^2}$ or $\frac{1}{2}\ln\frac{4+n^2}{8(n-1)^2}$

Q12e $\frac{\left(z^2-1\right)z^{-1}}{\left(z^2+1\right)z^{-1}}=\frac{z-z^{-1}}{z+z^{-1}}=\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}=\frac{i\,2\sin\theta}{2\cos\theta}=i\tan\theta$ is purely imaginary.

Q13a Contrapositive: If n can be even, then there is an integer $n\geq 3$ such that 2^n-1 is not prime.
Proof: Let $n=4$, $2^4-1=15$ which is not prime.

Q13b $a_n=2\cos\frac{\pi}{2^{n+1}}$

For $n=1$, $a_1=2\cos\frac{\pi}{4}=\sqrt{2}$ is true

For $n=k$, assume $a_k=2\cos\frac{\pi}{2^{k+1}}$ to prove that $a_{k+1}=2\cos\frac{\pi}{2^{k+2}}$.

Proof: Given $a_{n+1}^2 = 2 + a_n$.: $a_{n+1} = \sqrt{2 + a_n}$

For $n = k+1$, $a_{k+1} = \sqrt{2 + a_k} = \sqrt{2 + 2\cos\frac{\pi}{2^{k+1}}} = \sqrt{2\left(1 + \cos\frac{\pi}{2^{k+1}}\right)}$

$= \sqrt{2\left(2\cos^2\frac{\pi}{2\times 2^{k+1}}\right)} = \sqrt{2\left(2\cos^2\frac{\pi}{2^{k+2}}\right)} = 2\cos\frac{\pi}{2^{k+2}}$

.: $a_n = 2\cos\frac{\pi}{2^{n+1}}$ is true for all integers $n \ge 1$.

Q13ci $z^5 = -1 = \text{cis}(2k+1)\pi$, $z = \text{cis}\left(\frac{2k+1}{5}\right)\pi$ where

$k = 0, \pm 1, \pm 2$.: $z = -1,\ \text{cis}\left(\pm\frac{\pi}{5}\right),\ \text{cis}\left(\pm\frac{3\pi}{5}\right)$ are the 5th roots of -1.

Q13cii

$z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1) = (z+1)z^2\left(z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2}\right)$

$= (z+1)z^2\left(\left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) - 1\right) = 0$

Since $z \ne 0$ and $z \ne -1$,

.: $\left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) - 1 = 0$.: $u^2 - u - 1 = 0$

Q13ciii Let $z = \text{cis}\frac{3\pi}{5}$, $u = z + \frac{1}{z} = 2\cos\frac{3\pi}{5} < 0$

$u^2 - u - 1 = 0$.: $u = \frac{1-\sqrt{5}}{2}$.: $\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$

Q14ai Prove the contrapositive:

If $\lambda \ne 0$ and $\mu \ne 0$, $\lambda\vec{u} + \mu\vec{v} = \vec{0}$, $\vec{v} = -\frac{\lambda}{\mu}\vec{u}$, i.e. $\vec{u} \,//\, \vec{v}$

Q14aii Non-parallel $\vec{u}$ and $\vec{v}$, and $(\lambda_1 - \lambda_2)\vec{u} + (\mu_1 - \mu_2)\vec{v} = \vec{0}$

From part ai, $\lambda_1 - \lambda_2 = 0$ and $\mu_1 - \mu_2 = 0$.: $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$

Q14aiii Let L divides BC such that $BL : LC = m : n$

.: $\overrightarrow{SL} = \frac{n\overrightarrow{SB} + m\overrightarrow{SC}}{m+n}$, $\overrightarrow{SL} = k\overrightarrow{SK} = \frac{k}{4}\overrightarrow{SB} + \frac{k}{3}\overrightarrow{SC}$

.: $\frac{n}{m+n} = \frac{k}{4}$ and $\frac{m}{m+n} = \frac{k}{3}$.: $m : n = 4 : 3$

.: $\overrightarrow{BL} = \frac{4}{4+3}\overrightarrow{BC} = \frac{4}{7}\overrightarrow{BC}$

Q14aiv $\overrightarrow{AP} = -\left(6\overrightarrow{AB} + 8\overrightarrow{AC}\right) = -14\left(\frac{3}{7}\overrightarrow{AB} + \frac{4}{7}\overrightarrow{AC}\right) = -14\overrightarrow{AL}$

.: $AP \,//\, AL$ with common point A .: points P, A and L are collinear

.: P lies on AL.

Q14bi $J_0 = \int_0^1 x^0 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}$

Q14bii $0 < e^{-x} \le 1$ for $0 \le x \le 1$

.: $J_n = \int_0^1 x^n e^{-x}\,dx \le \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$, $J_n \le \frac{1}{n+1}$

Q14biii Integration by parts for $n \ge 1$:

$J_n = \int_0^1 x^n e^{-x}\,dx = \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx = -\frac{1}{e} + nJ_{n-1}$

Q14biv For $n = 0$, $J_0 = 0! - \frac{0!}{e}\sum_{r=0}^{0}\frac{1}{r!} = 1 - \frac{1}{e}$ is true.

For $n = k$, assume $J_k = k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}$

For $n = k+1$,

$J_{k+1} = (k+1)\left(k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right) - \frac{1}{e} = (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{1}{e}$

$= (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!} + \frac{(k+1)!}{e}\times\frac{1}{(k+1)!} - \frac{1}{e}$

.: $J_{k+1} = (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}$ is true

Hence $J_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}$ is true for all $n \ge 0$.

Q14bv Since $0 \le x \le 1$, $x^n e^{-x} \ge 0$.: $J_n \ge 0$

$0 \le J_n \le \frac{1}{n+1}$, $0 \le n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!} \le \frac{1}{n+1}$

$0 \le 1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!} \le \frac{1}{(n+1)n!}$, $0 \le 1 - \frac{1}{e}\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!} \le \lim_{n\to\infty}\frac{1}{(n+1)n!}$

$0 \le 1 - \frac{1}{e}\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!} \le 0$, .: $e = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}$

Q15ai Horizontal: $T_2\cos\phi - T_1\cos\theta = 0$

Vertical: $T_1\sin\theta - T_2\sin\phi - Mg = 0$

Eliminate T_1: $\frac{T_2\cos\phi}{\cos\theta}\sin\theta = T_2\sin\phi + Mg$

Hence $\tan\theta = \tan\phi + \frac{Mg}{T_2\cos\phi}$

Q15aii Let point P be y metres above the ground .: P is $h - y$ metres below the ceiling.

Since $\frac{Mg}{T_2\cos\phi} > 0$.: $\tan\phi < \tan\theta$.: $\frac{y}{2d} < \frac{h-y}{d}$.: $y < \frac{2h}{3}$

With the machine attached, $y < \frac{2h}{3}$.

Note: This seems to contradict the assumption in the first paragraph of part a, 'at all times the machine moves vertically upwards at a **constant velocity**' which $\tan\theta = \tan\phi + \frac{Mg}{T_2\cos\phi}$ is based.

Q15b SHM: $\ddot{x} = -n^2(x - c)$, $\frac{1}{40} = \frac{2\pi}{n}$, $n = 80\pi$,

$c = \frac{0.17 + 0.05}{2} = 0.11$, $\ddot{x} = -n^2(x - 0.11)$

Max acceleration at either extreme positions $x = 0.05$ or 0.17

Resultant force $= m\ddot{x} = -0.8(80\pi)^2(0.05 - 0.11) \approx 3032$ N

Q15c $x=\tan^2\theta$, $0\le x\le 1$, $0\le\theta\le\frac{\pi}{4}$, $\tan^2\theta=\sec^2\theta-1$

$dx=2\tan\theta\sec^2\theta\,d\theta$, $\sqrt{\frac{x}{1+x}}=\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}=\sin\theta$

$$\int_0^1\sin^{-1}\sqrt{\frac{x}{1+x}}\,dx=\int_0^{\frac{\pi}{4}}2\theta\tan\theta\sec^2\theta\,d\theta$$

$=\left[\theta\tan^2\theta\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan^2\theta\,d\theta$ integration by parts

$=\frac{\pi}{4}-\left[\tan\theta-\theta\right]_0^{\frac{\pi}{4}}=\frac{\pi}{2}-1$

Q15d $|z|=\left|z-\frac{4}{z}+\frac{4}{z}\right|\le\left|z-\frac{4}{z}\right|+\left|\frac{4}{z}\right|$ $\therefore |z|\le 2+\frac{4}{|z|}$

$\therefore |z|^2-2|z|-4\le 0$, $|z|^2-2|z|+1-5\le 0$, $(|z|-1)^2\le 5$

$\therefore -\sqrt{5}\le|z|-1\le\sqrt{5}$, $1-\sqrt{5}\le|z|\le 1+\sqrt{5}$ $\therefore |z|\le\sqrt{5}+1$

Q16a $z_A=5+i$, let $z_B=b+5i$ and $z_C=c-5i$

Rotate z_B about z_A anticlockwise by $\frac{\pi}{3}$ to reach z_C.

$(z_B-z_A)e^{i\frac{\pi}{3}}=(z_C-z_A)$, $((b-5)+4i)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=(c-5)-6i$

Equate imaginary parts: $b=5-\frac{16}{\sqrt{3}}$ $\therefore z_B=5-\frac{16}{\sqrt{3}}+5i$

Q16b $\vec{R}=M\vec{g}+0.1M\vec{v}$, $\frac{d\vec{v}}{dt}=\vec{g}+0.1\vec{v}=-10+0.1\vec{v}$

Initially upward motion, scalar resolute $\frac{dv}{dt}=-10-0.1v$

$\frac{dt}{dv}=-\frac{1}{0.1}\times\frac{1}{100+v}$, $-0.1t=\int\frac{1}{100+v}dv$, $-0.1t=\ln(100+v)+c$

$t=0$, $v=v_0$ $\therefore -0.1t=\ln\frac{100+v}{100+v_0}$ $\therefore v=\frac{dx}{dt}=(100+v_0)e^{-0.1t}-100$

Displacement is zero when the projectile returns to the same point at

$t=7$ $\therefore \int_0^7\left((100+v_0)e^{-0.1t}-100\right)dt=0$, $\left[-\frac{100+v_0}{0.1}e^{-0.1t}-100t\right]_0^7=0$

$-\frac{100+v_0}{0.1}e^{-0.7}-700+\frac{100+v_0}{0.1}=0$, $(100+v_0)\left(1-e^{-0.7}\right)=70$

$v_0\approx 39.1$

Q16ci $V=abc$, $S=2ab+2bc+2ca$

Given $\frac{x_1x_2x_3}{\left(\frac{x_1+x_2+x_3}{3}\right)^3}\le 1$, $x_1x_2x_3\le\left(\frac{x_1+x_2+x_3}{3}\right)^3$

Let $x_1=2ab$, $x_2=2bc$ and $x_3=2ca$

$8(abc)^2\le\frac{S^3}{27}$ $\therefore (abc)^2\le\frac{S^3}{6^3}$, $abc\le\left(\frac{S}{6}\right)^{\frac{3}{2}}$

Q16cii $V\le\left(\frac{S}{6}\right)^{\frac{3}{2}}$

When the rectangular prism is a cube, $a=b=c$,

$V=a^3$ and $\left(\frac{S}{6}\right)^{\frac{3}{2}}=\left(\frac{6a^2}{6}\right)^{\frac{3}{2}}=a^3$

Since $V=a^3=\left(\frac{S}{6}\right)^{\frac{3}{2}}$, the prism has maximum volume of a^3.

Q16d $z=re^{i\theta}$, $|z_1|=|z_2|=|z_3|=r$

$z_1z_2z_3=1$, $|z_1||z_2||z_3|=1$ $\therefore r^3=1$, $r=1$

Consider z_1, z_2 and z_3 as roots of $P(z)$

$P(z)=(z-z_1)(z-z_2)(z-z_3)=0$,

$z^3-(z_1+z_2+z_3)z^2+(z_2z_3+z_3z_1+z_1z_2)z-z_1z_2z_3=0$

$\therefore z^3-1z^2+z_1z_2z_3(\bar{z}_1+\bar{z}_2+\bar{z}_3)z-1=0$

$z^3-1z^2+1(1)z-1=0$ (Note: $\bar{z}_1+\bar{z}_2+\bar{z}_3=\overline{z_1+z_2+z_3}=1$)

$(z-1)(z^2+1)=0$, $z=1,\pm i$

Please inform mathline@itute.com re conceptual and/or mathematical errors.

