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Specialist Mathematics

2022

Trial Examination 2 (2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ m s}^{-2}$, where $g = 9.8$

Question 1 $\int_0^{\frac{1}{a}} \frac{1}{\pi} \cos^{-1}(ax) dx =$

A. $\frac{a}{\pi}$

B. $\frac{\pi}{a}$

C. $\frac{1}{a\pi}$

D. $\frac{2}{a\pi}$

E. $\frac{a\pi}{4}$

Question 2 Given vectors $a\tilde{i} + b\tilde{j}$ and $c\tilde{j} + d\tilde{k}$ make 60° angle, then $3b^2c^2 =$

A. $c^2a^2 + a^2d^2 + d^2b^2$

B. $c^2a^2 - a^2d^2 + d^2b^2$

C. $c^2a^2 + a^2d^2 - d^2b^2$

D. $c^2a^2 - a^2d^2 - d^2b^2$

E. $-c^2a^2 + a^2d^2 + d^2b^2$

Question 3 A 0.01-kg particle is projected at 5.0 m s^{-1} vertically upwards. The distance (m) travelled in the time interval $[0.25, 0.75]$ in seconds is closest to

A. 0.3

B. 0.6

C. 0.9

D. 1.2

E. 1.5

Question 4 Let $z = \text{cis}\left(\frac{(n+3)\pi}{11}\right)$ and $\text{Arg}(z^k) = \text{Arg}(z)$ where $n, k \in \mathbb{Z}^+ \setminus \{1\}$, the smallest value of k is

- A. 6
- B. 5
- C. 4
- D. 3
- E. 2

Question 5 A particle is in equilibrium when acted on by three coplanar forces \vec{F}_1 , \vec{F}_2 and \vec{F}_3 . The angle between \vec{F}_1 and \vec{F}_2 , and the angle between \vec{F}_2 and \vec{F}_3 are 135° and 105° respectively.

The value of $\frac{|\vec{F}_3|}{|\vec{F}_2|}$ is closest to

- A. 2.01
- B. 1.90
- C. 1.64
- D. 1.08
- E. 0.82

Question 6 Vector $\tilde{c} = p\tilde{i} + q\tilde{j} + r\tilde{k}$ is linearly independent of $\tilde{a} = \tilde{i} - \tilde{j}$ and $\tilde{b} = \tilde{j} + \tilde{k}$ for $p, q, r \in \mathbb{R}$, then

- A. $p + q = 0$ and $q + r = 0$
- B. $p + r = 0$ and $q + r = 0$
- C. $p + q = 0$ and $q - r = 0$
- D. $p + r = 0$ and $q - r = 0$
- E. $p - r = 0$ and $q - r = 0$

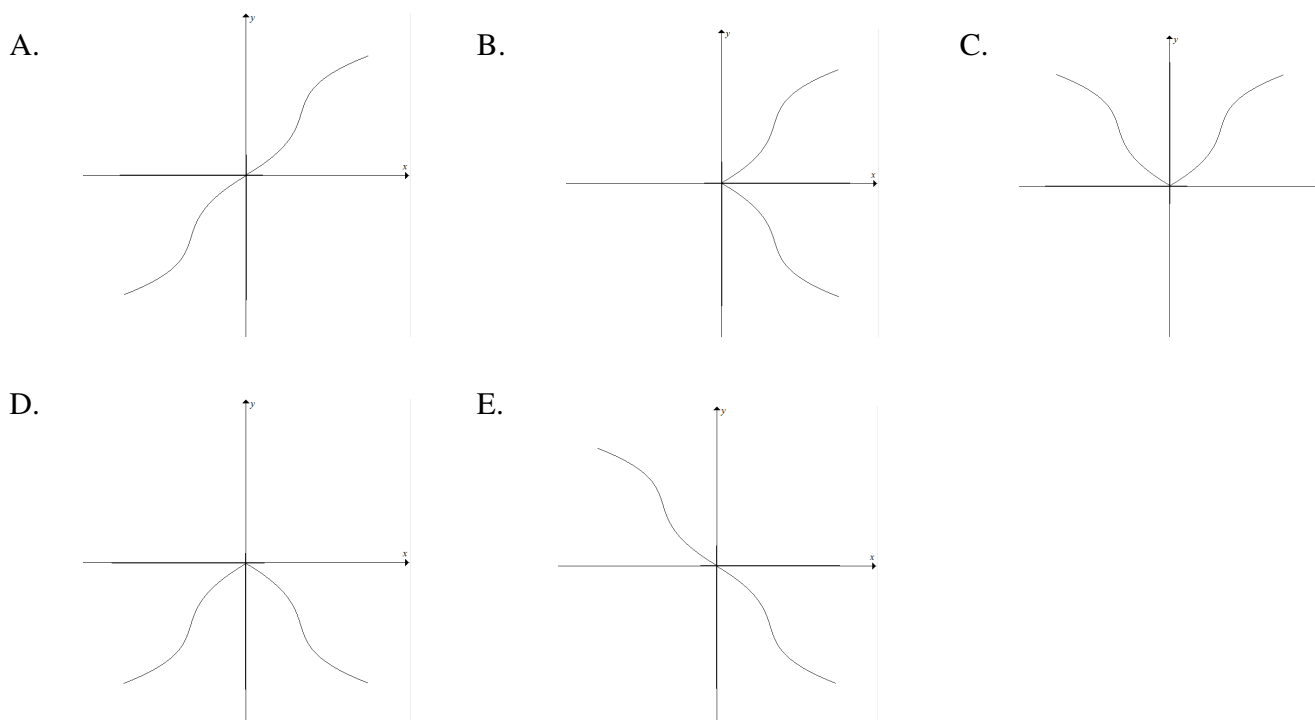
Question 7 Given $z = x + yi = \text{cis } \theta$ where $\theta = \text{Arg}(z)$, then $z + 1 =$

- A. $\sqrt{\frac{1-y}{2}} \text{cis}(2\theta)$
- B. $\sqrt{\frac{1+x}{2}} \text{cis}(2\theta)$
- C. $\sqrt{2(1+x)} \text{cis}\left(\frac{\theta}{2}\right)$
- D. $\sqrt{2(1-y)} \text{cis}\left(\frac{\theta}{2}\right)$
- E. $\sqrt{\frac{1+y}{1+x}} \text{cis}\left(\frac{\theta}{2}\right)$

Question 8 The imaginary part of a possible cube root of $z = -i \operatorname{cis}\left(-\frac{\pi}{4}\right)$ is

- A. $\frac{\sqrt{3}-1}{2\sqrt{2}}$
- B. $\frac{1-\sqrt{3}}{2\sqrt{2}}$
- C. $\frac{1-\sqrt{2}}{2\sqrt{3}}$
- D. $\frac{\sqrt{2}-1}{2\sqrt{3}}$
- E. $\frac{\sqrt{2}-\sqrt{3}}{2\sqrt{6}}$

Question 9 $f(x)$ is an odd function and $f(-1) < 0$. A possible graph showing $y = f(-|x|)$ is



Question 10 Let $a, b \in \mathbb{R}^+$, $z_1 = -a + bi$ and $z_2 = b + ai$, $\frac{z_1}{iz_2} =$

- A. 1
- B. 0.5
- C. i
- D. $0.5i$
- E. $1-i$

Question 11 Point $\left(\frac{\pi}{2}, 0\right)$ is on the curve defined by $\frac{dy}{dx} = \frac{\sin(x)}{\cos(y)}$. The curve is given by

- A. $\sin(y) - \cos(x) = 0$
- B. $\sin(y) + \cos(x) = 0$
- C. $\sin(y)\cos(x) = 0$
- D. $\sin(y) - \cos(x) = 0, x \in R \setminus \{0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots\}$
- E. $\sin(y) + \cos(x) = 0, x \in R \setminus \{0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots\}$

Question 12 A particle has position vector given by $\tilde{r}(t) = \cos^{-1}(t)\tilde{i} + \sin^{-1}(t)\tilde{j}, t \in [-1, 1]$. The distance travelled by the particle in the interval $[-1, 1]$ is closest to

- A. 6
- B. $\sqrt{3}\pi$
- C. 5
- D. $\sqrt{2}\pi$
- E. 4

Question 13 A particle moves in a straight line and its position is given by x .

It starts from rest at $x = -1$ and accelerates at $-\sqrt{v^2 + 1}$ where v is its velocity. Its velocity at $x = 2$ is

- A. $-\sqrt{5}$
- B. $\sqrt{5}$
- C. $-\sqrt{3}$
- D. $\sqrt{3}$
- E. undefined

Question 14 The scalar resolute of $-\tilde{i} + 2\tilde{j}$ in the direction of $2\tilde{i} - \tilde{j} + 2\tilde{k}$ is

- A. -4
- B. 4
- C. $-\frac{4}{3}$
- D. $\frac{4}{3}$
- E. undefined

Question 15 Given $\frac{dy}{dx} = f(x)$. The value of $\int_1^{1.01} f(x)dx$ is closest to

- A. $0.01f(1)$
- B. $0.01f(1.01)$
- C. $1.01f(1.01)$
- D. $1.01f(1)$
- E. $1.01f(0.99)$

Question 16 Given $-1 < \tan x < 0$, which one of the following statements **cannot** be true?

- A. $\tan^2 \frac{x}{2} < 1 + 2 \tan \frac{x}{2} < 1$
- B. $1 - \sqrt{2} < \tan \frac{x}{2} < 0$
- C. $0 < \tan \frac{x}{2} < 1 + \sqrt{2}$
- D. $\tan \frac{x}{2} > 1 + \sqrt{2}$
- E. $\tan \frac{x}{2}$ can have a positive or negative value

Question 17 The curve $x^2 + (y-1)^2 = 1$ for $y \geq 1$ is rotated about the x -axis. The volume of the solid of revolution formed is V where

- A. $V = \frac{4\pi}{3} \left[1 - \left(\sqrt{1-x^2} \right)^3 \right]_{-1}^1$
- B. $V = \frac{4\pi}{3} \left[1 + \left(\sqrt{1-x^2} \right)^3 \right]_{-1}^1$
- C. $V = \frac{8\pi}{3} \left[\left(1 + \sqrt{1-x^2} \right)^3 \right]_0^1$
- D. $V = 2\pi \left(\frac{5}{3} + 2 \int_0^1 \sqrt{1-x^2} dx \right)$
- E. $V = 4\pi \left(\frac{5}{3} + \int_0^1 \sqrt{1-x^2} dx \right)$

Question 18 Given random variables X and Y where $Y = 2X - 1$, for $a, b \in \mathbb{R}^+$ and $b > a$, $E(aX - bY)$ and $\text{sd}(aX - bY)$ are respectively

- A. $aE(X) - bE(Y)$ and $\sqrt{a^2 \text{Var}(X) + b^2 \text{Var}(Y)}$
- B. $aE(X) - bE(Y)$ and $\sqrt{a^2 \text{Var}(X) - b^2 \text{Var}(Y)}$
- C. $aE(X) + bE(Y)$ and $\sqrt{a^2 \text{Var}(X) + b^2 \text{Var}(Y)}$
- D. $(a - 2b)E(X) + b$ and $(2b - a)\text{sd}(X)$
- E. $(2b - a)E(X) + b$ and $(2b - a)\text{sd}(X)$

Question 19 Random variable X in a large population is normally distributed.

A random sample of size 100 is taken and an approximate 80% confidence interval for μ of X is calculated to be (16.8, 17.2).

The sample standard deviation is closest to

- A. 0.92
- B. 1.09
- C. 1.13
- D. 1.17
- E. 1.55

Question 20 The mean of random variable X of a very large population is 175.

20 out of 50 random samples of size 100 from the population have their mean values of X greater than 178.

The standard deviation of X is closest to

- A. 122
- B. 118
- C. 105
- D. 104
- E. 103

SECTION B Extended-answer questions

Instructions for Section B

Answer **all** questions.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ m s}^{-2}$, where $g = 9.8$

Question 1 (9 marks)

Let $z^2 = (2 + \sqrt{2})\text{cis}\left(-\frac{3\pi}{4}\right)\text{cis}\left(-\frac{\pi}{12}\right)\text{cis}\left(\frac{7\pi}{12}\right)$, and $w = r\text{cis}\theta$ where $r = |w|$ and $\theta = \text{Arg}(w)$.

a. Find r and θ such that $wz \in \mathbb{R}^-$ and $|wz| = \sqrt{2}$.

3 marks

b. Find z in $a + bi$ form.

3 marks

c. $0 + 0i$ is a member of the set complex numbers $v = x + yi$ equidistant from z^2 .

Show that the other members satisfy $\frac{x^2 + y^2}{2(x + y)} = 1 + \sqrt{2}$.

3 marks

Question 2 (16 marks)

At time t (min), the amount of salt in the vat is Q grams.

The vat has 10 litres of solution containing 1 gram of salt initially ($t = 0$).

At time t , a solution of salt concentration kQ grams per litre runs into the vat at $\frac{20}{10+t}$ litres per min, $k \in \mathbb{R}^+$.

The mixture is stirred and drained at 1 litre per min.

a. State the rate of inflow of salt into the vat at time t in terms of k , t and Q . Include units. 1 mark

b. Show that the salt concentration in the vat at time t is $\frac{(10+t)Q}{100+20t-t^2}$. Include units. 2 marks

c. State the rate of outflow of salt from the vat at time t in terms of t and Q . Include units. 1 mark

d. Write a differential equation for the rate of change of the amount of salt in the vat. Include units. 1 mark

e. State the time interval that your equation in part d is valid correct to the nearest min. 1 mark

f. Given $100 + 20t - t^2 = (\alpha - t)(\beta + t)$, $\alpha, \beta \in R^+$, show (NOT verify) that $\alpha = 10\sqrt{2} + 10$ and $\beta = 10\sqrt{2} - 10$.

2 marks

g. Show that $\frac{10+t}{100+20t-t^2} = \frac{A}{10\sqrt{2}+10-t} + \frac{B}{10\sqrt{2}-10+t}$ where $A = \frac{\sqrt{2}+1}{2}$ and $B = \frac{\sqrt{2}-1}{2}$. 2 marks

h. Solve the differential equation in part d to show

$\log_e Q = 20k \log_e \left(\frac{10+t}{10} \right) + \frac{\sqrt{2}+1}{2} \log_e \left(\frac{10\sqrt{2}+10-t}{10\sqrt{2}+10} \right) - \frac{\sqrt{2}-1}{2} \log_e \left(\frac{10\sqrt{2}-10+t}{10\sqrt{2}-10} \right)$ for $0 \leq t \leq 24$. 4 marks

i. Find the value of k (correct to 2 decimal places) such that $Q \approx 0.5639$ at $t = 15$, and the time (correct to the nearest min) when the amount of salt in the vat decreases at maximum rate.

2 marks

Question 3 (14 marks)

The positions of Particle A and Particle B at time $t \geq 0$ is given by

$$\tilde{r}_A(t) = (\sin t)\tilde{i} + (2\cos t)\tilde{j} \quad \text{and} \quad \tilde{r}_B(t) = (1 + \sin t)\tilde{i} + (\cos t)\tilde{j} \quad \text{respectively.}$$

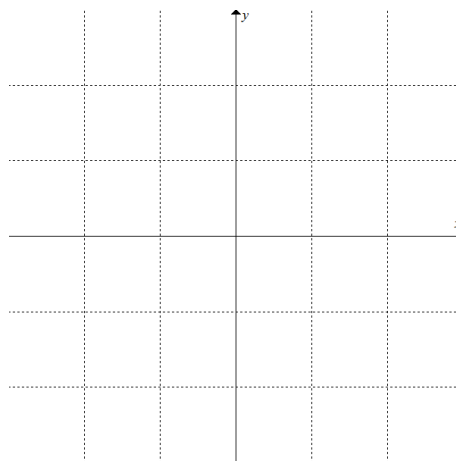
\tilde{i} points in the positive x -direction and \tilde{j} in the positive y -direction.

a. Determine the Cartesian equation of the path for each particle.

2 marks

b. Sketch the paths of the particles on the Cartesian plane shown below. Label each path with A or B . Dot the initial position of each particle. Draw arrow on the path to show the direction of each particle's motion.

3 marks



c. Find the minimum time required for each particle to return to its initial position.

1 mark

d. Show that the distance D separating the two particles is $D = \sqrt{1 + \cos^2 t}$ at time t .

1 mark

Let $n = 0, 1, 2, 3, \dots$.

e. Determine the shortest distance and the longest distance between the two particles, and the times (in terms of n) when they occur.

2 marks

f. Find the velocity of each particle in terms of t .

Hence, express the time in terms of n when the particles move in the same direction.

2 marks

g. Let θ be the angle between the two velocity vectors. Express $\cos \theta$ in terms of $\sin t$.

Find the largest angle (in radian correct to 2 decimal places) between the two velocity vectors.

3 marks

Question 4 (12 marks)

A 1 kg particle is projected up an inclined plane at time $t = 0$ and its velocity is $v = V_0$ at position $x = 0$.
 x is measured along the plane.

The angle between the inclined plane and the horizontal is 30° .

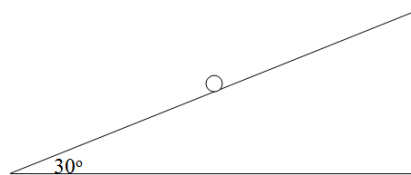
It experiences a resistive force of kv^2 where k is a real constant.

The force of friction between the particle and the inclined plane is 1 newton.

Time is measured in seconds, length in metres and force in newtons.

a. Draw a labeled diagram showing the forces pointing in the correct direction and their magnitude (newtons) on the particle during its upward motion at time t .

2 marks



b. Write down the equation of motion for the upward motion of the particle on the plane.

1 mark

c. Find the relation between x and v by solving a differential equation for the motion of the particle on the plane, and the maximum displacement from $x = 0$ reached by the particle in terms of k and V_0 .

3 marks

d. Find the time taken by the particle to reach its maximum displacement from $x = 0$ in terms of k and V_0 .

2 marks

The particle experiences the same resistive force during its motion down the inclined plane.

Take $t = 0$ when the particle starts to slide down the plane.

e. Find the relation between t and v by solving a differential equation for the downward motion of the particle, and the terminal velocity reached by the particle in terms of k .

4 marks

Question 5 (9 marks)

Correct answers to 3 significant figures unless stated otherwise.

Nails are made by two machines, Machine *A* and Machine *B*, in a factory.

The length X cm of a nail from each machine has a normal distribution.

Machine *A* produces nails of mean length 5.22 cm and standard deviation of 0.150 cm.

Machine *B* produces nails of mean length 5.40 cm and the same standard deviation of 0.150 cm.

Machine *A* produces twice as many nails as Machine *B* daily.

- a. Show that the mean length and standard deviation of the nails produced by the factory in a day are 5.28 and 0.1118 cm respectively.

2 marks

- b. Calculate the probability that a nail produced by the factory in a day is longer than 5.30 cm.

1 mark

50 random samples of size 100 are taken from the nails produced by the factory in a day.

- c. How many (to the nearest unit) samples have a mean length shorter than 5.27 cm?

1 mark

- d. Given that $\Pr(5.28 - a < \bar{X} < 5.28 + a) = 0.90$, find the value of a , correct to 2 decimal places.

2 marks

A new machine, Machine *C*, is installed in the factory. It produces a large number of nails in a day.

It is calibrated and expected to produce nails of mean length 5.30 cm.

The length X cm of a nail from Machine *C* has a normal distribution.

A random sample of 10 nails from Machine *C* is taken, and the length (cm) of each nail is measured:

5.31, 5.33, 5.28, 5.32, 5.30, 5.32, 5.33, 5.30, 5.32, 5.30

- e. Find the mean length and standard deviation of the nails in the sample.

1 mark

- f. Determine the approximate 95% confidence interval for the mean length of the nails produced by Machine *C* in a day.

1 mark

- g. Comment on whether Machine *C* needs to be recalibrated.

1 mark

End of Exam 2