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2022 Specialist Mathematics Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
C	A	B	E	E	B	C	B	C	A	E	D	E	C	A	D	D	D	E	B

Question 1 $\int_0^{\frac{1}{a}} \frac{1}{\pi}\cos^{-1}(ax)\,dx = \int_0^{\frac{1}{2}} x\,dy = \int_0^{\frac{1}{2}} \frac{1}{a}\cos \pi y\,dy = \frac{1}{a\pi}$ C. $\frac{1}{a\pi}$

Question 2 Given vectors $a\tilde{i}+b\tilde{j}$ and $c\tilde{j}+d\tilde{k}$ make $60°$ angle, then $3b^2c^2 =$

$\left(a\tilde{i}+b\tilde{j}\right).\left(c\tilde{j}+d\tilde{k}\right)=\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos 60°$, $bc=\frac{1}{2}\sqrt{a^2c^2+a^2d^2+b^2c^2+b^2d^2}$,

$4b^2c^2 = a^2c^2+a^2d^2+b^2c^2+b^2d^2$ A. $c^2a^2+a^2d^2+d^2b^2$

Question 3 A 0.01-kg particle is projected at 5.0 m s^{-1} vertically upwards. The distance (m) travelled in the time interval $[0.25, 0.75]$ in seconds is closest to

The particle reaches the highest point at approximately 0.5 s. Distance travelled from 0.25 s to the top is approx. 0.3 m, and from the top to position at 0.75 s is also approx. 0.3 m. B. 0.6

Question 4 Let $z=\text{cis}\left(\frac{(n+3)\pi}{11}\right)$ and $\text{Arg}\left(z^k\right)=\text{Arg}(z)$ where $n,k\in Z^+\setminus\{1\}$, the smallest value of k is

$\text{Arg}\left(z^k\right)=\text{Arg}(z)$, $\frac{k(n+3)\pi}{11}-2\pi m=\frac{(n+3)\pi}{11}$ where $m\in Z$

Smallest $k=1+\frac{22m}{n+3}=2$ when $m=1$ and $n=19$ E. 2

Question 5 A particle is in equilibrium when acted on by three coplanar forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$. The angle between $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$, and the angle between $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ are $135°$ and $105°$ respectively.

The value of $\frac{\left|\overrightarrow{F_3}\right|}{\left|\overrightarrow{F_2}\right|}$ is closest to

$\frac{\left|\overrightarrow{F_3}\right|}{\sin 45°}=\frac{\left|\overrightarrow{F_2}\right|}{\sin 60°}$, $\frac{\left|\overrightarrow{F_3}\right|}{\left|\overrightarrow{F_2}\right|}=\frac{\sqrt{2}}{\sqrt{3}}$ E. 0.82

Question 6 Vector $\tilde{c}=p\tilde{i}+q\tilde{j}+r\tilde{k}$ is linearly independent of $\tilde{a}=\tilde{i}-\tilde{j}$ and $\tilde{b}=\tilde{j}+\tilde{k}$ for $p,q,r\in R$, then a component of $\tilde{c}$ is perpendicular to $\tilde{a}$ and a component of $\tilde{c}$ is perpendicular to $\tilde{b}$

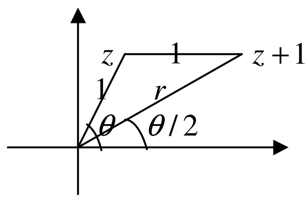
$\therefore \left(p\tilde{i}+q\tilde{j}\right).\left(\tilde{i}-\tilde{j}\right)=0$, $p-q=0$ and $\left(q\tilde{j}+r\tilde{k}\right).\left(\tilde{j}+\tilde{k}\right)=0$, $q+r=0$

$\therefore p+r=0$ B. $p+r=0$ and $q+r=0$

Question 7 Given $z=x+yi=\text{cis}\,\theta$ where $\theta=\text{Arg}(z)$, then $z+1=$

$r^2=1^2+1^2-2\cos(\pi-\theta)=2+2\cos\theta=2+2x$, $r=\sqrt{2(1+x)}$

$z+1=r\,\text{cis}\left(\frac{\theta}{2}\right)=\sqrt{2(1+x)}\,\text{cis}\left(\frac{\theta}{2}\right)$ C. $\sqrt{2(1+x)}\,\text{cis}\left(\frac{\theta}{2}\right)$



Question 8 The imaginary part of a possible cube root of $z=-i\,\text{cis}\left(-\frac{\pi}{4}\right)$ is

$z=-i\,\text{cis}\left(-\frac{\pi}{4}\right)=\text{cis}\left(-\frac{\pi}{2}\right)\text{cis}\left(-\frac{\pi}{4}\right)=\text{cis}\left(-\frac{3\pi}{4}\right)=\text{cis}\left(2k-\frac{3}{4}\right)\pi$ where $k\in Z$

$z^{\frac{1}{3}}=\text{cis}\left(-\frac{\pi}{4}\right)$ or $\text{cis}\left(\frac{5\pi}{12}\right)$ or $\text{cis}\left(-\frac{11\pi}{12}\right)$ for $k=0,1,-1$ respectively. $\sin\left(-\frac{11\pi}{12}\right)=\frac{1-\sqrt{3}}{2\sqrt{2}}$ B. $\frac{1-\sqrt{3}}{2\sqrt{2}}$

Question 9 $f(x)$ is an odd function and $f(-1)<0$. A possible graph showing $y=f(-|x|)$ is C.

Question 10 Let $a,\,b\in R^{+}$, $z_1=-a+bi$ and $z_2=b+ai$, $\frac{z_1}{i\,z_2}=$

$i\,z_2=-a+bi$ $\therefore\ \frac{z_1}{i\,z_2}=1$ A. 1

Question 11 Point $\left(\frac{\pi}{2},0\right)$ is on the curve defined by $\frac{dy}{dx}=\frac{\sin(x)}{\cos(y)}$. The curve is given by

$\frac{dy}{dx}=\frac{\sin(x)}{\cos(y)}$ where $\sin(x)\neq 0$ and $\cos(y)\neq 0$, $\int\cos(y)dy=\int\sin(x)dx$

Since $\left(\frac{\pi}{2},0\right)$ $\therefore\ \sin(y)=-\cos(x)$, $\sin(y)+\cos(x)=0$, and since $\sin(x)\neq 0$ $\therefore\ x\in R\setminus\{0,\pm\pi,\pm 2\pi,\pm 3\pi,\cdots\}$

E. $\sin(y)+\cos(x)=0$, $x\in R\setminus\{0,\pm\pi,\pm 2\pi,\pm 3\pi,\cdots\}$

Question 12 A particle has position vector given by $\tilde{r}(t)=\cos^{-1}(t)\tilde{i}+\sin^{-1}(t)\tilde{j}$, $t\in[-1,1]$.
The distance travelled by the particle in the interval $[-1,1]$ is closest to

$x'(t)=\frac{-1}{\sqrt{1-t^2}}$, $y'(t)=\frac{1}{\sqrt{1-t^2}}$, distance $=\int_{-1}^{1}\sqrt{\left(\frac{-1}{\sqrt{1-t^2}}\right)^2+\left(\frac{-1}{\sqrt{1-t^2}}\right)^2}\,dt=\sqrt{2}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}dt=\sqrt{2}\pi$ D. $\sqrt{2}\pi$

Question 13 A particle moves in a straight line and its position is given by x.
It starts from rest at $x=-1$ and accelerates at $-\sqrt{v^2+1}$ where v is its velocity. Its velocity at $x=2$ is

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-\sqrt{v^2+1}$, $2\frac{dx}{d(v^2)}=-\frac{1}{(v^2+1)^{\frac{1}{2}}}$, $2x=-2\sqrt{v^2+1}+c$, $v=0$ at $x=-1$

$\therefore\ x=-\sqrt{v^2+1}$ i.e. $x\in R^{-}$ E. undefined

Question 14 The scalar resolute of $-\tilde{i}+2\tilde{j}$ in the direction of $2\tilde{i}-\tilde{j}+2\tilde{k}$ is

$\tilde{a}=2\tilde{i}-\tilde{j}+2\tilde{k}$, $\tilde{b}=-\tilde{i}+2\tilde{j}$, $\tilde{b}.\hat{a}=-\frac{4}{3}$ C. $-\frac{4}{3}$

Question 15 Given $\frac{dy}{dx}=f(x)$. The value of $\int_{1}^{1.01}f(x)dx$ is closest to

Let $y=c$ when $x=1$ $\therefore\ y=\int_{1}^{1.01}f(x)dx+c$ when $x=1.01$

Linear approximation $y\approx c+0.01f(1)$ $\therefore\ \int_{1}^{1.01}f(x)dx\approx 0.01f(1)$ A. $0.01f(1)$

Question 16 Given $-1<\tan x<0$, which one of the following statements **cannot** be true?

$-1<\tan x<0$, $-1<\dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}<0$, $-1+\tan^2\dfrac{x}{2}<2\tan\dfrac{x}{2}<0$, $\tan^2\dfrac{x}{2}<1+2\tan\dfrac{x}{2}<1$,

$\tan^2\dfrac{x}{2}-2\tan\dfrac{x}{2}-1<0$, $\left(\tan\dfrac{x}{2}-1\right)^2-2<0$, $1-\sqrt{2}<\tan\dfrac{x}{2}<1+\sqrt{2}$ D. $\tan\dfrac{x}{2}>1+\sqrt{2}$

Question 17 The curve $x^2+(y-1)^2=1$ for $y\ge 1$ is rotated about the x-axis.
The volume of the solid of revolution formed is V where

$y=\sqrt{1-x^2}+1$, $V=2\int_0^1 \pi y^2 dx=2\pi\int_0^1\left(\sqrt{1-x^2}+1\right)^2 dx=2\pi\int_0^1\left(2-x^2+2\sqrt{1-x^2}\right)dx$ D. $V=2\pi\left(\dfrac{5}{3}+2\int_0^1\sqrt{1-x^2}\,dx\right)$

Question 18 Given random variables X and Y where $Y=2X-1$,
for $a,b\in R^+$ and $b>a$, $\mathrm{E}(aX-bY)$ and $\mathrm{sd}(aX-bY)$ are respectively

$Y=2X-1$, $aX-bY=aX-b(2X-1)=(a-2b)X+b$

$\mathrm{E}(aX-bY)=\mathrm{E}((a-2b)X+b)=(a-2b)\mathrm{E}(X)+b$

$\mathrm{Var}(aX-bY)=\mathrm{Var}((a-2b)X+b)=(a-2b)^2\mathrm{Var}(X)=(2b-a)^2\mathrm{Var}(X)$ D. $(a-2b)\mathrm{E}(X)+b$ and $(2b-a)\mathrm{sd}(X)$

Question 19 Random variable X in a large population is normally distributed.
A random sample of size 100 is taken and an approximate 80% confidence interval for μ of X is calculated to be $(16.8, 17.2)$.
The sample standard deviation is closest to

For 80% confidence interval, $z\approx 1.28$, $\bar{x}=17$, $17+1.28\times\dfrac{s}{\sqrt{100}}\approx 17.2$, $s\approx 1.56$ E. 1.55

Question 20 The mean of random variable X of a very large population is 175.
20 out of 50 random samples of size 100 from the population have their mean values of X greater than 178.
The standard deviation of X is closest to

$\Pr(\bar{X}>178)=\dfrac{20}{50}$, $\Pr\left(Z>\dfrac{178-175}{\mathrm{sd}(\bar{X})}\right)=\dfrac{20}{50}$, $\mathrm{sd}(\bar{X})\approx 11.84$, $\dfrac{\sigma}{\sqrt{100}}\approx 11.84$, $\sigma\approx 118$ B. 118

SECTION B Extended-answer questions

Question 1 (9 marks)

Let $z^2=\left(2+\sqrt{2}\right)\mathrm{cis}\left(-\dfrac{3\pi}{4}\right)\mathrm{cis}\left(-\dfrac{\pi}{12}\right)\mathrm{cis}\left(\dfrac{7\pi}{12}\right)$, and $w=r\,\mathrm{cis}\,\theta$ where $r=|w|$ and $\theta=\mathrm{Arg}(w)$.

a. Find r and θ such that $wz\in R^-$ and $|wz|=\sqrt{2}$. 3 marks

$z^2=\left(2+\sqrt{2}\right)\mathrm{cis}\left(2k\pi-\dfrac{\pi}{4}\right)$, $z=\sqrt{2+\sqrt{2}}\,\mathrm{cis}\left(k\pi-\dfrac{\pi}{8}\right)=\sqrt{2+\sqrt{2}}\,\mathrm{cis}\left(-\dfrac{\pi}{8}\right)$ or $\sqrt{2+\sqrt{2}}\,\mathrm{cis}\left(\dfrac{7\pi}{8}\right)$

$|wz|=\sqrt{2}$, $r\sqrt{2+\sqrt{2}}=\sqrt{2}$, $r=\dfrac{\sqrt{2}}{\sqrt{2}\left(\sqrt{2}+1\right)}=\dfrac{1}{\sqrt{2}+1}=\sqrt{2}-1$

$wz\in R^-$ $\therefore$ $-\dfrac{\pi}{8}+\theta=-\pi$ or $\dfrac{7\pi}{8}+\theta=\pi$ $\therefore$ $\theta=-\dfrac{7\pi}{8}$ or $\dfrac{\pi}{8}$

b. Find z in $a+bi$ form. 3 marks

$$\cos\frac{\pi}{4}=\cos 2\left(\frac{\pi}{8}\right),\ \frac{1}{\sqrt{2}}=2\cos^2\frac{\pi}{8}-1,\ \cos^2\frac{\pi}{8}=\frac{2+\sqrt{2}}{4},\ \cos\frac{\pi}{8}=\frac{\sqrt{2+\sqrt{2}}}{2}\ \therefore\ \sin\frac{\pi}{8}=1-\cos^2\frac{\pi}{8}=\frac{1}{\sqrt{2}\sqrt{2+\sqrt{2}}}$$

$$z=\sqrt{2+\sqrt{2}}\operatorname{cis}\left(-\frac{\pi}{8}\right)=\sqrt{2+\sqrt{2}}\left(\cos\left(-\frac{\pi}{8}\right)+i\sin\left(-\frac{\pi}{8}\right)\right)=\sqrt{2+\sqrt{2}}\left(\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}\right)=\left(1+\frac{1}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}i$$

or $\sqrt{2+\sqrt{2}}\operatorname{cis}\left(\frac{7\pi}{8}\right)=\sqrt{2+\sqrt{2}}\left(\cos\frac{7\pi}{8}+i\sin\frac{7\pi}{8}\right)=\sqrt{2+\sqrt{2}}\left(-\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)=-\left(1+\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}i$

c. $0+0i$ is a member of the set complex numbers $v=x+yi$ equidistant from z^2.

Show that the other members satisfy $\frac{x^2+y^2}{2(x+y)}=1+\sqrt{2}$. 3 marks

$z^2=\left(2+\sqrt{2}\right)\operatorname{cis}\left(-\frac{\pi}{4}\right)=\left(1+\sqrt{2}\right)-\left(1+\sqrt{2}\right)i$ Since $0+0i$ is a member of the set complex numbers $v=x+yi$

equidistant from z^2, the distance is $\left|z^2\right|=\sqrt{\left(1+\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)^2}=2+\sqrt{2}$

$\therefore\ \left|v-z^2\right|^2=\left(2+\sqrt{2}\right)^2,\ x^2-2\left(1+\sqrt{2}\right)(x+y)+y^2=0,\ \frac{x^2+y^2}{2(x+y)}=1+\sqrt{2}$

Question 2 (16 marks)

At time t (min), the amount of salt in the vat is Q grams.
The vat has 10 litres of solution containing 1 gram of salt initially ($t=0$).
At time t, a solution of salt concentration kQ grams per litre runs into the vat at $\frac{20}{10+t}$ litres per min, $k\in R^+$.
The mixture is stirred and drained at 1 litre per min.

a. State the rate of inflow of salt into the vat at time t in terms of k, t and Q. Include units. 1 mark

Rate of inflow $=\frac{20kQ}{10+t}$ grams per minute

b. Show that the salt concentration in the vat at time t is $\frac{(10+t)Q}{100+20t-t^2}$. Include units. 2 marks

Volume $V=10+\left(\frac{20}{10+t}-1\right)t=\frac{100+20t-t^2}{10+t}$ litres at time t

Concentration $=\frac{Q}{V}=\frac{(10+t)Q}{100+20t-t^2}$ grams per litre

c. State the rate of outflow of salt from the vat at time t in terms of t and Q. Include units. 1 mark

Rate of outflow $= \frac{(10+t)Q}{100+20t-t^2}$ grams per minute

d. Write a differential equation for the rate of change of the amount of salt in the vat. Include units. 1 mark

$\frac{dQ}{dt} = \frac{20kQ}{10+t} - \frac{(10+t)Q}{100+20t-t^2}$ grams per minute

e. State the time interval that your equation in part d is valid correct to the nearest min. 1 mark

$\frac{100+20t-t^2}{10+t} \geq 0$, $0 \leq t \leq 24$

f. Given $100+20t-t^2 = (\alpha - t)(\beta + t)$, $\alpha, \beta \in R^+$, show (NOT verify) that $\alpha = 10\sqrt{2}+10$ and $\beta = 10\sqrt{2}-10$. 2 marks

$100+20t-t^2 = 200-(t^2-20t+100) = (10\sqrt{2})^2-(t-10)^2 = (10\sqrt{2}-t+10)(10\sqrt{2}+t-10) = (\alpha-t)(\beta+t)$

$\therefore$ $\alpha = 10\sqrt{2}+10$ and $\beta = 10\sqrt{2}-10$

g. Show that $\frac{10+t}{100+20t-t^2} = \frac{A}{10\sqrt{2}+10-t} + \frac{B}{10\sqrt{2}-10+t}$ where $A = \frac{\sqrt{2}+1}{2}$ and $B = \frac{\sqrt{2}-1}{2}$. 2 marks

$\frac{10+t}{100+20t-t^2} = \frac{A}{10\sqrt{2}+10-t} + \frac{B}{10\sqrt{2}-10+t}$, $10+t = A(10\sqrt{2}-10+t) + B(10\sqrt{2}+10-t)$

Let $t = 10\sqrt{2}+10$, $A = \frac{\sqrt{2}+1}{2}$; let $t = -10\sqrt{2}+10$, $B = \frac{\sqrt{2}-1}{2}$

$\therefore$ $\frac{10+t}{100+20t-t^2} = \frac{\frac{\sqrt{2}+1}{2}}{10\sqrt{2}+10-t} + \frac{\frac{\sqrt{2}-1}{2}}{10\sqrt{2}-10+t}$

h. Solve the differential equation in part d to show

$\log_e Q = 20k\log_e\left(\frac{10+t}{10}\right) + \frac{\sqrt{2}+1}{2}\log_e\left(\frac{10\sqrt{2}+10-t}{10\sqrt{2}+10}\right) - \frac{\sqrt{2}-1}{2}\log_e\left(\frac{10\sqrt{2}-10+t}{10\sqrt{2}-10}\right)$ for $0 \leq t \leq 24$. 4 marks

$\frac{dQ}{dt} = \frac{20kQ}{10+t} - \frac{(10+t)Q}{100+20t-t^2} = \left(\frac{20k}{10+t} - \frac{10+t}{100+20t-t^2}\right)Q$

$\int \frac{1}{Q}dQ = \int\left(\frac{20k}{10+t} - \frac{10+t}{100+20t-t^2}\right)dt$, $\int \frac{1}{Q}dQ = \int\left(\frac{20k}{10+t} - \frac{\frac{\sqrt{2}+1}{2}}{10\sqrt{2}+10-t} - \frac{\frac{\sqrt{2}-1}{2}}{10\sqrt{2}-10+t}\right)dt$

$\log_e Q = 20k\log_e(10+t) + \frac{\sqrt{2}+1}{2}\log_e(10\sqrt{2}+10-t) - \frac{\sqrt{2}-1}{2}\log_e(10\sqrt{2}-10+t) + c$

$Q = 1$ when $t = 0$ $\therefore$ $c = -20k\log_e 10 - \frac{\sqrt{2}+1}{2}\log_e(10\sqrt{2}+10) + \frac{\sqrt{2}-1}{2}\log_e(10\sqrt{2}-10)$

$\therefore$ $\log_e Q = 20k\log_e\left(\frac{10+t}{10}\right) + \frac{\sqrt{2}+1}{2}\log_e\left(\frac{10\sqrt{2}+10-t}{10\sqrt{2}+10}\right) - \frac{\sqrt{2}-1}{2}\log_e\left(\frac{10\sqrt{2}-10+t}{10\sqrt{2}-10}\right)$

i. Find the value of k (correct to 2 decimal places) such that $Q \approx 0.5639$ at $t = 15$, and the time (correct to the nearest min) when the amount of salt in the vat decreases at maximum rate.

2 marks

By CAS
$Q \approx 0.5639$ at $t = 15$.: $k \approx 0.05$
Decreases at maximum rate at $t \approx 20$

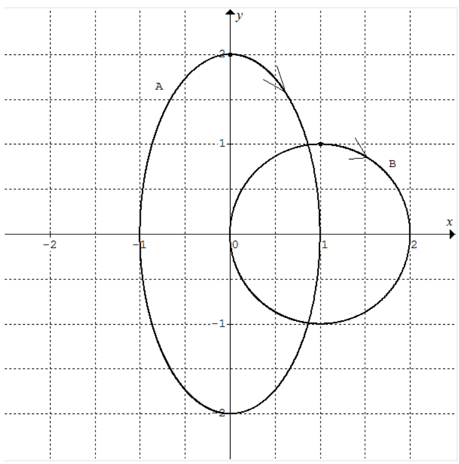
Question 3 (14 marks)
The positions of Particle A and Particle B at time $t \geq 0$ is given by
$\underset{\sim}{r}_A(t) = (\sin t)\underset{\sim}{i} + (2\cos t)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (1+\sin t)\underset{\sim}{i} + (\cos t)\underset{\sim}{j}$ respectively.
$\underset{\sim}{i}$ points in the positive x-direction and $\underset{\sim}{j}$ in the positive y-direction.

a. Determine the Cartesian equation of the path for each particle. 2 marks

For A: $x = \sin t$, $y = 2\cos t$; $x^2 + \left(\frac{y}{2}\right)^2 = 1$
For B: $x = 1 + \sin t$, $y = \cos t$; $(x-1)^2 + y^2 = 1$

b. Sketch the paths of the particles on the Cartesian plane shown below. Label each path with A or B. Dot the initial position of each particle. Draw arrow on the path to show the direction of each particle's motion.

3 marks



c. Find the minimum time required for each particle to return to its initial position. 1 mark

For A: $2\cos t = 2$, $t = 0, 2\pi$.: minimum time $= 2\pi - 0 = 2\pi$
For B: $\cos t = 1$, $t = 0, 2\pi$.: minimum time $= 2\pi$

d. Show that the distance D separating the two particles is $D = \sqrt{1+\cos^2 t}$ at time t. 1 mark

$\underset{\sim}{r}_B - \underset{\sim}{r}_A = \underset{\sim}{i} - (\cos t)\underset{\sim}{j}$, $D = |\underset{\sim}{r}_B - \underset{\sim}{r}_A| = \sqrt{1+\cos^2 t}$

Let $n = 0, 1, 2, 3, \cdots$.

e. Determine the shortest distance and the longest distance between the two particles, and the times (in terms of n) when they occur. 2 marks

Shortest $D = 1$ when $\cos^2 t = 0$, $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots = \left(n + \frac{1}{2}\right)\pi$

Longest $D = \sqrt{2}$ when $\cos^2 t = 1$, $\cos t = \pm 1$, $t = n\pi$

f. Find the velocity of each particle in terms of t.
Hence, express the time in terms of n when the particles move in the same direction. 2 marks

$\tilde{v}_A = (\cos t)\tilde{i} - (2\sin t)\tilde{j}$, $\tilde{v}_B = (\cos t)\tilde{i} - (\sin t)\tilde{j}$

In the same direction when $t = n\pi$ and $\tilde{v}_A = \tilde{v}_B = \tilde{i}$ or when $t = \left(n + \frac{1}{2}\right)\pi$ and $\tilde{v}_A = \pm 2\tilde{j}$, $\tilde{v}_B = \pm \tilde{j}$

g. Let θ be the angle between the two velocity vectors. Express $\cos\theta$ in terms of $\sin t$.
Find the largest angle (in radian correct to 2 decimal places) between the two velocity vectors. 3 marks

$\tilde{v}_A . \tilde{v}_B = |\tilde{v}_A||\tilde{v}_B|\cos\theta$, $\cos^2 t + 2\sin^2 t = \sqrt{\cos^2 t + 4\sin^2 t}\sqrt{\cos^2 t + \sin^2 t}\cos\theta$, $1 + \sin^2 t = \sqrt{1 + 3\sin^2 t}\cos\theta$

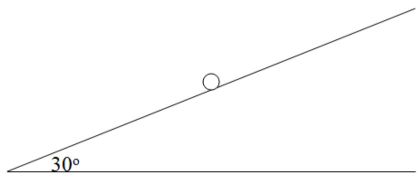
$\therefore$ $\cos\theta = \frac{1 + \sin^2 t}{\sqrt{1 + 3\sin^2 t}}$, by CAS, sketch $\theta = \cos^{-1}\frac{1 + \sin^2 t}{\sqrt{1 + 3\sin^2 t}}$ and find $\theta \approx 0.34$

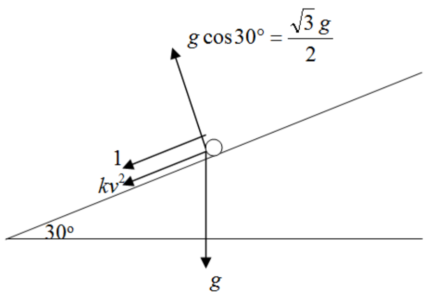
Question 4 (12 marks)

A 1 kg particle is projected up an inclined plane at time $t = 0$ and its velocity is $v = V_0$ at position $x = 0$.
x is measured along the plane.
The angle between the inclined plane and the horizontal is $30°$.
It experiences a resistive force of kv^2 where k is a real constant.
The force of friction between the particle and the inclined plane is 1 newton.
Time is measured in seconds, length in metres and force in newtons.

a. Draw a labeled diagram showing the forces pointing in the correct direction and their magnitude (newtons) on the particle during its upward motion at time t.

2 marks





b. Write down the equation of motion for the upward motion of the particle on the plane. 1 mark

$a = \frac{R}{m}$, $a = \frac{-1 - 0.5g - kv^2}{1} = -(5.9 + kv^2)$

c. Find the relation between x and v by solving a differential equation for the motion of the particle on the plane, and the maximum displacement from $x=0$ reached by the particle in terms of k and V_0.

3 marks

$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=-\left(5.9+kv^2\right)$, $\frac{1}{2k}\frac{d\left(v^2\right)}{dx}=-\left(\frac{5.9}{k}+v^2\right)$.: $-2kx=\int\frac{1}{\frac{5.9}{k}+v^2}d\left(v^2\right)$

$-2kx=\log_e\left(\frac{5.9}{k}+v^2\right)+c$ where $c=-\log_e\left(\frac{5.9}{k}+V_0^{\,2}\right)$.: $-2kx=\log_e\left(\frac{\frac{5.9}{k}+v^2}{\frac{5.9}{k}+V_0^{\,2}}\right)$

Max displacement is reached when $v=0$.: maximum displacement $=x_{\max}=\frac{1}{2k}\log_e\left(1+\frac{k}{5.9}V_0^{\,2}\right)$

d. Find the time taken by the particle to reach its maximum displacement from $x=0$ in terms of k and V_0.

2 marks

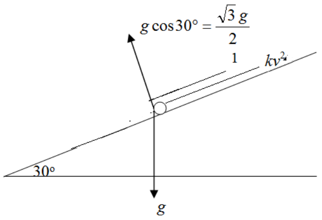
$\frac{dv}{dt}=-\left(5.9+kv^2\right)=-k\left(\frac{5.9}{k}+v^2\right)$, $\frac{dt}{dv}=-\frac{1}{k\left(\frac{5.9}{k}+v^2\right)}$

.: time taken to reach max displacement ($v=0$): $t=-\frac{1}{k}\int_{V_0}^{0}\frac{1}{\frac{5.9}{k}+v^2}dv=\frac{1}{\sqrt{5.9k}}\tan^{-1}\left(\sqrt{\frac{k}{5.9}}V_0\right)$

The particle experiences the same resistive force during its motion down the inclined plane.
Take $t=0$ when the particle starts to slide down the plane.

e. Find the relation between t and v by solving a differential equation for the downward motion of the particle, and the terminal velocity reached by the particle in terms of k.

4 marks



Take down the inclined plane as the positive direction.

$a=0.5g-1-kv^2=k\left(\frac{3.9}{k}-v^2\right)$, $\frac{dv}{dt}=k\left(\frac{3.9}{k}-v^2\right)$, $\frac{dt}{dv}=\frac{1}{k\left(\frac{3.9}{k}-v^2\right)}$

$kt=\int\frac{1}{\left(\frac{3.9}{k}-v^2\right)}dv=\frac{1}{2}\sqrt{\frac{k}{3.9}}\int\left(\frac{1}{\sqrt{\frac{3.9}{k}}-v}-\frac{1}{\sqrt{\frac{3.9}{k}}+v}\right)dv$

$=\frac{1}{2}\sqrt{\frac{k}{3.9}}\left(-\log_e\left(\sqrt{\frac{3.9}{k}}-v\right)-\log_e\left(\sqrt{\frac{3.9}{k}}+v\right)\right)+c=-\frac{1}{2}\sqrt{\frac{k}{3.9}}\log_e\left(\frac{3.9}{k}-v^2\right)+c$

At $t=0$, $v=0$.: $c=\frac{1}{2}\sqrt{\frac{k}{3.9}}\log_e\left(\frac{3.9}{k}\right)$.: $t=\frac{1}{2\sqrt{3.9k}}\log_e\left(\frac{\frac{3.9}{k}}{\frac{3.9}{k}-v^2}\right)$

As $t\to\infty$, $\frac{3.9}{k}-v^2\to 0$, $v\to\sqrt{\frac{3.9}{k}}$, the terminal velocity

Question 5 (9 marks)
Correct answers to 3 significant figures unless stated otherwise.
Nails are made by two machines, Machine *A* and Machine *B*, in a factory.
The length X cm of a nail from each machine has a normal distribution.
Machine *A* produces nails of mean length 5.22 cm and standard deviation of 0.150 cm.
Machine *B* produces nails of mean length 5.40 cm and the same standard deviation of 0.150 cm.
Machine *A* produces twice as many nails as Machine *B* daily.

a. Show that the mean length and standard deviation of the nails produced by the factory in a day are 5.28 and 0.1118 cm respectively.

2 marks

$$\mu = \frac{2}{3}\times 5.22 + \frac{1}{3}\times 5.40 = 5.28 \quad \text{Var}(X) = \left(\frac{2}{3}\right)^2 0.15^2 + \left(\frac{1}{3}\right)^2 0.15^2 = 0.0125 \quad \sigma = \sqrt{0.0125} \approx 0.112$$

b. Calculate the probability that a nail produced by the factory in a day is longer than 5.30 cm. 1 mark

By CAS, $\Pr(X > 5.30) \approx 0.429$

50 random samples of size 100 are taken from the nails produced by the factory in a day.

c. How many (to the nearest unit) samples have a mean length shorter than 5.27 cm? 1 mark

$\text{E}(\overline{X}) = \mu = 5.28$, $\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}} \approx 0.0112$, by CAS $\Pr(\overline{X} < 5.27) \approx 0.1860$, $0.1860 \times 50 \approx 9$

d. Given that $\Pr(5.28 - a < \overline{X} < 5.28 + a) = 0.90$, find the value of a, correct to 2 decimal places. 2 marks

$\Pr\left(-\frac{a}{0.0112} < Z < \frac{a}{0.0112}\right) = 0.90$, $\Pr\left(Z < \frac{a}{0.0112}\right) = 0.95$, $a \approx 0.018 \approx 0.02$

A new machine, Machine *C*, is installed in the factory. It produces a large number of nails in a day.
It is calibrated and expected to produce nails of mean length 5.30 cm.
The length X cm of a nail from Machine *C* has a normal distribution.
A random sample of 10 nails from Machine *C* is taken, and the length (cm) of each nail is measured:
5.31, 5.33, 5.28, 5.32, 5.30, 5.32, 5.33, 5.30, 5.32, 5.30

e. Find the mean length and standard deviation of the nails in the sample. 1 mark

$\bar{x} = 5.311 \approx 5.31$, $s_x \approx 0.01595 \approx 0.0160$

f. Determine the approximate 95% confidence interval for the mean length of the nails produced by Machine *C* in a day.

1 mark

$$\left(5.311 - 1.96\times\frac{0.0160}{\sqrt{10}},\ 5.311 + 1.96\times\frac{0.0160}{\sqrt{10}}\right) \approx \left(5.30^+, 5.32^+\right)$$

g. Comment on whether Machine *C* needs to be recalibrated. 1 mark

Since $5.30 \notin (5.30, 5.32)$, Machine *C* needs to be recalibrated at 5% level of significance.

