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2022 Specialist Mathematics

Year 12 Application Task

(Time allowed: 4 hours plus)

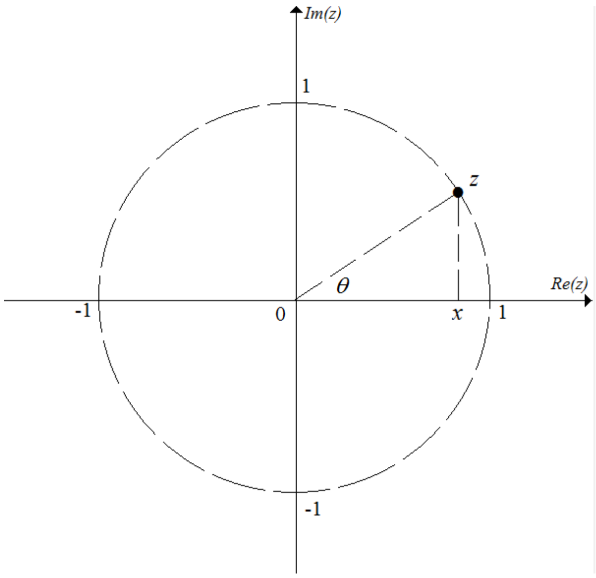
Theme: Chebyshev polynomials

The Chebyshev polynomials are a sequence of polynomials related to de Moivre's formula.
In this task you are to generate the Chebyshev polynomials and explore some of their properties and applications.

Assumed knowledge: Complex numbers, functions/graphs, compound/double angle formulas, identities, recursion, differentiation, integration, CAS

Part I (80-90 min)

Consider complex number z with argument θ shown in the following Argand diagram.



a. Express z in polar form.

b. State $\bar{z}$ and plot it accurately on the above Argand diagram.

c. Plot z^2 and $\bar{z}^2$ accurately on the above Argand diagram.

d. Express $\frac{1}{2}\left(z^2 + \bar{z}^2\right)$ in terms of θ.

e. Express $\frac{1}{2}\left(z^n + \bar{z}^n\right)$ in terms of θ, $n = 0, 1, 2, 3, \cdots$.

f. Hence show that $\frac{1}{2}\left(z^n+\bar{z}^n\right)=\cos\left(n\cos^{-1}(x)\right)$ where $x=\text{Re}(z)$.

Let $T_n(x)=\cos\left(n\cos^{-1}(x)\right)$.

g. State the domain and range of $T_n(x)$.

h. Show that $T_0(x)=1$ and $T_1(x)=x$.

i. By means of the double angle and compound angle formulas, show that $T_2(x)=2x^2-1$ and $T_3(x)=4x^3-3x$.

$T_1(x)$, $T_2(x)$ etc are called Chebyshev polynomials. $T_n(x)$ is the n^{th} Chebyshev polynomial.
For $n=3, 4, \cdots$, $T_n(x)$ can be determined by repeated use of the compound angle formula, and the process is very laborious.

j. Given $T_n(x)=\cos\left(n\cos^{-1}(x)\right)$, write an expression for each of $T_{n+1}(x)$ and $T_{n-1}(x)$.

k. Show that $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$.

l. Use the information in part h, part i and the recurrence relation in part k, show that Chebyshev polynomials $T_4(x)=8x^4-8x^2+1$ and $T_5(x)=16x^5-20x^3+5x$.

Four more Chebyshev polynomials of higher degrees are shown below.

$T_6(x)=32x^6-48x^4+18x^2-1$

$T_7(x)=64x^7-112x^5+56x^3-7x$

$T_8(x)=128x^8-256x^6+160x^4-32x^2+1$

$T_9(x)=256x^9-576x^7+432x^5-120x^3+9x$

m. Study the coefficient of the first term of each Chebyshev polynomial discussed so far.
In terms of n write down the coefficient of the first term of Chebyshev polynomial T_n.

n. The sum of coefficients equals 1 in each of the nine Chebyshev polynomials investigated above.
Prove that the sum of coefficients equals 1 in any $T_n(x)$.

End of Part I

Part II (80-90 min)

If students attempt Part II at a latter time, they should have copies of their Part I to access the given information and their workings.

$T_0(x)$ and the first nine Chebyshev polynomials are reprinted below for reference.

$T_0(x)=1$
$T_1(x)=x$
$T_2(x)=2x^2-1$
$T_3(x)=4x^3-3x$
$T_4(x)=8x^4-8x^2+1$
$T_5(x)=16x^5-20x^3+5x$
$T_6(x)=32x^6-48x^4+18x^2-1$
$T_7(x)=64x^7-112x^5+56x^3-7x$
$T_8(x)=128x^8-256x^6+160x^4-32x^2+1$
$T_9(x)=256x^9-576x^7+432x^5-120x^3+9x$

where $T_n(x)=\cos\left(n\cos^{-1}(x)\right),\ x\in[-1,1]$.

a. Show (**NOT verify**) that or explain why $T_n(x)$ is a n^{th} degree polynomial.
Hint: Use the definition $T_n(x)=\cos\left(n\cos^{-1}(x)\right)$.

b. Show (**NOT verify**) that a general solution to the equation $T_n(x)=0$ is $x=\cos\frac{(2k+1)\pi}{2n}$ where $n\neq 0$ and $k\in Z$, set of integers.
Hint: Use the definition $T_n(x)=\cos\left(n\cos^{-1}(x)\right)$.

c. **Use part b** to find the solutions to $T_3(x)=0$.

d. Show that $\frac{d}{dx}T_n(x)=\frac{n\sin(n\theta)}{\sin\theta}$ where $\theta=\cos^{-1}(x)$.

e. **Use part d** to show that the *x*-coordinates of the stationary points of $T_n(x)$ is given by $x=\cos\left(\frac{k\pi}{n}\right)$ where $k\in Z$.

f. **Use part e** to find the exact *x*-coordinates of the stationary points of $T_6(x)$.

$T_n(x)=\cos\left(n\cos^{-1}(x)\right)$, replace n with $n+1$ and $n-1$ to obtain
$T_{n+1}(x)=\cos\left((n+1)\cos^{-1}(x)\right)$ and $T_{n-1}(x)=\cos\left((n-1)\cos^{-1}(x)\right)$.

g. Show that $\frac{1}{n+1}\frac{d}{dx}T_{n+1}(x)=\frac{\sin((n+1)\theta)}{\sin\theta}$ and $\frac{1}{n-1}\frac{d}{dx}T_{n-1}(x)=\frac{\sin((n-1)\theta)}{\sin\theta}$, where $\theta=\cos^{-1}(x)$.

h. Use the results in part g to show that $\frac{1}{n+1}\frac{d}{dx}T_{n+1}(x)-\frac{1}{n-1}\frac{d}{dx}T_{n-1}(x)=2T_n$.

Hint: Use the compound angle formula.

i. Use $T_2(x)$, $T_3(x)$ and $T_4(x)$ to verify the formula in part h.

j. Find $\int T_3(x)dx$ and $\frac{1}{2}\left(\frac{T_4(x)}{4}-\frac{T_2(x)}{2}\right)$. Compare the two results.

k. Use the formula in part h to find a formula for $\int T_n(x)dx$.

l. Name a property of Chebyshev polynomials $T_n(x)$ for $n = 1, 3, 5, \cdots$, which results in $\int_{-1}^{1} T_n(x)dx = 0$.

Explain in terms of area bounded by T_n, the x-axis, $x = -1$ and $x = 1$.

m. For $n = 2, 4, 6, \cdots$, find a formula for $\int_{-1}^{1} T_n(x)dx$ in terms of n.

End of Part II

Part III (80-90 min)

If students attempt Part III at a latter time, they should have copies of their Part I and Part II to access the given information and their workings.

$T_0(x)$ and the first nine Chebyshev polynomials are reprinted below for reference.

$T_0(x)=1$
$T_1(x)=x$
$T_2(x)=2x^2-1$
$T_3(x)=4x^3-3x$
$T_4(x)=8x^4-8x^2+1$
$T_5(x)=16x^5-20x^3+5x$
$T_6(x)=32x^6-48x^4+18x^2-1$
$T_7(x)=64x^7-112x^5+56x^3-7x$
$T_8(x)=128x^8-256x^6+160x^4-32x^2+1$
$T_9(x)=256x^9-576x^7+432x^5-120x^3+9x$

where $T_n(x)=\cos\left(n\cos^{-1}(x)\right),\ x\in[-1,1]$.

Let $\theta=\cos^{-1}(x),\ T_n(\cos\theta)=\cos(n\theta)$

a. State the domain and range of $T_n(\cos\theta)=\cos(n\theta)$.

b. Use compound angle formulas to prove the identity $\cos(A)-\cos(B)=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

Use the identity $\cos(A)-\cos(B)=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ to answer parts c and d.

c. Solve $T_4(\cos\theta)-T_2(\cos\theta)=0$ for θ.

d. Solve $T_5(\cos\theta)-T_3(\cos\theta)=0$ for θ.

e. By CAS solve each of the two equations $T_4(x)-T_2(x)=0$ and $T_5(x)-T_3(x)=0$ for x, i.e. find the x-coordinates of the intersections of $T_4(x)$ and $T_2(x)$, and the intersections of $T_5(x)$ and $T_3(x)$.

f. Compare the answers in parts c and d with the answers in part e. Comment.

g. Find a general solution to $T_{n+1}(\cos\theta)-T_{n-1}(\cos\theta)=0$ in terms of n and k where $k\in Z$.
Hence find a general solution to $T_{n+1}(x)-T_{n-1}(x)=0$ in terms of n and k.
State the relationship between n and k.

h. Similarly, find a general solution to $T_{n+1}(\cos\theta)-T_n(\cos\theta)=0$ in terms of n and k, and hence find a general solution to $T_{n+1}(x)-T_n(x)=0$ in terms of n and k. State the relationship between n and k.

i. Choose a value of $n \in \{3, 4, 5\}$ to verify the general solutions in part h.

The following parts explore two examples of using a series of Chebyshev polynomials to approximate functions, $f(x) \approx a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + \cdots + a_nT_n(x)$ as $n \to \infty$, and a_k is the coefficient of T_k.

j. Consider $\log_e(1+x) \approx a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + a_3T_3(x) + a_4T_4(x) + a_5T_5(x)$.

The coefficients are given by $a_n = \begin{cases} -\log_e 2, & n = 0 \\ \dfrac{-2(-1)^n}{n}, & n > 0 \end{cases}$

(i) Express $a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + a_3T_3(x) + a_4T_4(x) + a_5T_5(x)$ as a polynomial of degree 5 in simplest form.

(ii) Sketch the graphs of $y = \log_e(1+x)$ and $y = a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + a_3T_3(x) + a_4T_4(x) + a_5T_5(x)$ on the same axes for $x \in (-1, 1)$.

(iii) Comment on the closeness of the approximation to $\log_e(1+x)$, and the effects of increasing the number of consecutive terms in the series.

k. Consider $\pi \approx \alpha\left(\frac{T_1(\cos\theta)}{1} - \frac{T_3(\cos\theta)}{3} + \frac{T_5(\cos\theta)}{5} - \frac{T_7(\cos\theta)}{7} + \frac{T_9(\cos\theta)}{9}\right)$ where $\alpha \in R^+$ and $T_n(\cos\theta) = \cos(n\theta)$.

(i) Find the value of α (correct to 1 decimal place) such that

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \pi \, d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \alpha\left(\frac{T_1(\cos\theta)}{1} - \frac{T_3(\cos\theta)}{3} + \frac{T_5(\cos\theta)}{5} - \frac{T_7(\cos\theta)}{7} + \frac{T_9(\cos\theta)}{9}\right) d\theta .$$

(ii) Sketch the graphs of $y = \pi$ and $y = \alpha\left(\frac{T_1(\cos\theta)}{1} - \frac{T_3(\cos\theta)}{3} + \frac{T_5(\cos\theta)}{5} - \frac{T_7(\cos\theta)}{7} + \frac{T_9(\cos\theta)}{9}\right)$ for $\theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ on the same axes.

(iii) Comment on the closeness of the two graphs, and the effects of increasing the number of consecutive terms in the series on the ripples.

End of Task

