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2022
Specialist Mathematics

Year 12
Problem Solving Task

(Time allowed: 2.0 hours plus)

Problem Solving Task

Theme: Oscillations

In this task you are to investigate factors affecting the period of an oscillation.

Assumed knowledge:

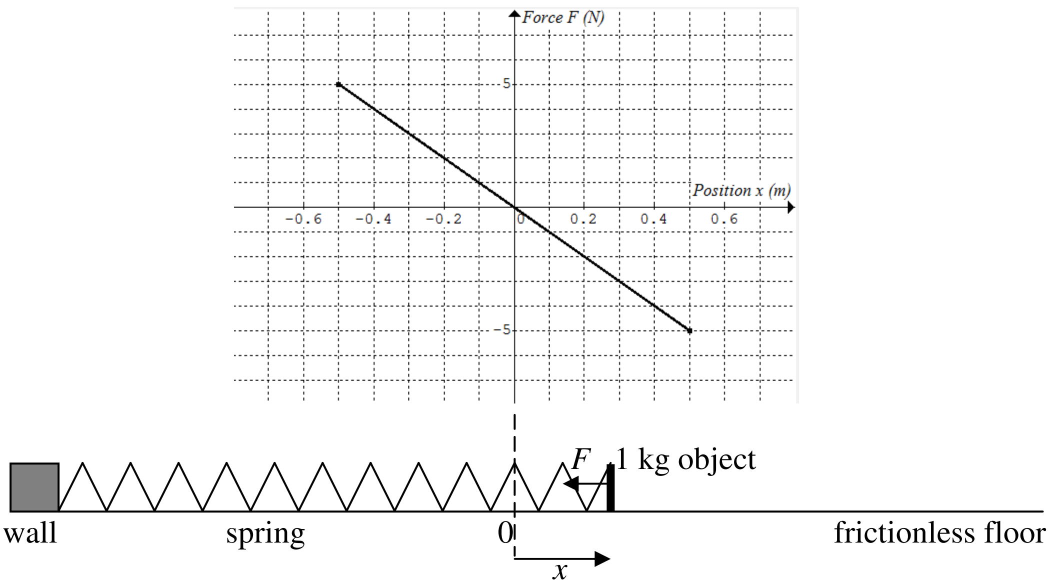
Newton's laws, functions and graphs, calculus, differential equations, use of CAS

Force is measured in newtons (N), distance in metres (m), time in seconds (s) and mass in kilograms (kg).

Assumptions:
(1) The springs are dimensionless and have no mass.
(2) The object is a dimensionless particle.
(3) Motions are frictionless and without air resistance unless stated otherwise.
(4) g is $9.8\,N\,kg^{-1}$.

Part I (60 – 65 minutes)

The behaviour of a compressible and extendable spring is shown in the following graph.
A 1 kg object is attached to the spring. It is pulled to position $x = 0.4$ m and released.



a. Show that the force F of the spring on the 1 kg object is given by $F = -10x$.

b. Write down the equation of motion of the object.

c. Solve the equation of motion to show that $\frac{dx}{dt} = -\sqrt{1.6 - 10x^2}$.
Explain/interpret the negative sign when x changes from 0.4 to 0.

d. Find the maximum speed of the object.

e. Solve $\frac{dx}{dt} = -\sqrt{1.6 - 10x^2}$ for x in terms of t.

f. Describe the motion in terms of position, velocity and acceleration.

g. Write down the period of the motion (oscillation).

An object of mass m kg is attached to another compressible and extendable spring of equation $F = -kx$, $x \in [-\beta, \beta]$ where F and x are defined as previously, $\beta \in R$ and k is a constant called spring constant. The object is pulled to position $x = \alpha$ where $0 < \alpha < \beta$, and released.

h. In terms of parameters m, k and α, write down and solve the equation of motion of the object to find the position $x(t)$, the period T and velocity $v(t)$.

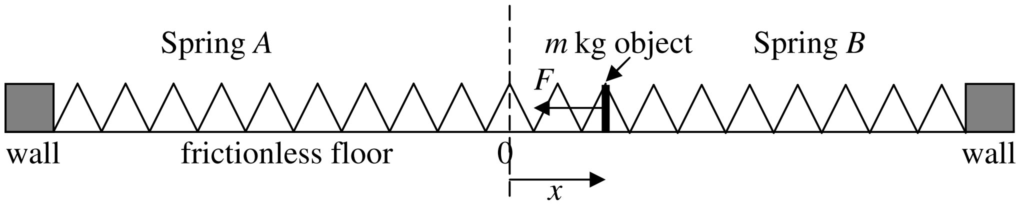
i. Does the period depend on the amplitude α of the motion (oscillation)? Explain.

j. Briefly discuss the effects of changing the object's mass m on the period of the motion.

k. Briefly discuss the effects of changing the spring constant k on the period of the motion.

l. Verify your answer to part k by letting $m=1$ and select a suitable value of α, sketch the graph of one cycle of $x(t)$ for 3 different values of k in the interval $(8,16)$. Comment

Spring A and Spring B have the same natural length.
A m kg object is attached to Spring A and Spring B as shown in the diagram below.
$F_A = -k_A x$ and $F_B = -k_B x$ for $x \in [-\beta, \beta]$, where k_A and k_B are the spring constants.



The object is pulled to position $x=\alpha$ and released, $0<\alpha<\beta$.
F is the resultant force (net force) on the object when it is at position x.

m. Write down the equation of motion of the m kg object in terms of m, α, k_A and k_B.

n. Find the position $x(t)$, the period T and velocity $v(t)$ of the m kg object in terms of m, α, k_A and k_B.

o. Does the period depend on the amplitude α of each motion (oscillation) investigated above ? Justify your answer.

End of Part I

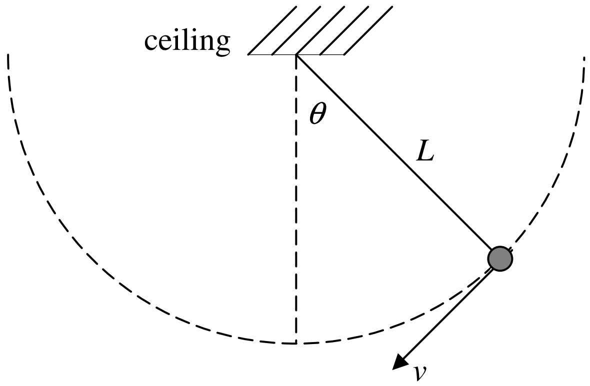
Part II (60 – 65 minutes)

Consider another oscillating system outlined below.

Force is measured in newtons (N), distance in metres (m), time in seconds (s) and mass in kilograms (kg).

Assumptions:
(1) The cord is dimensionless and has no mass.
(2) The object is a dimensionless particle.
(3) Motions are frictionless and without air resistance unless stated otherwise.
(4) g is $9.8\, N\, kg^{-1}$.

A m kg object is attached to a L m long cord fastened to the ceiling.
The object is pulled to the right until the cord makes angle α (amplitude of oscillation) with the vertical and then released from rest at $t = 0$.
When the cord makes angle θ with the vertical at time t, the speed of the object is $|v|$ m s^{-1}.
θ has a negative value on the left side of the vertical dotted line.



Speed is given by $|v| = L\left|\frac{d\theta}{dt}\right|$, and using energy consideration, $v^2 = 2gL(\cos\theta - \cos\alpha)$, $\theta \in [-\alpha, \alpha]$.

a. Show that $\frac{d\theta}{dt} = -\sqrt{\frac{2g}{L}(\cos\theta - \cos\alpha)}$.
Explain/interpret the negative sign when θ changes from α to 0.

b. Show that $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$.

c. Sketch the graph of $y=\frac{\sin\theta}{\theta}$ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and find the value of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$.

d. Explain why $\frac{d^2\theta}{dt^2}=-\frac{g}{L}\theta$ is an approximation to $\frac{d^2\theta}{dt^2}=-\frac{g}{L}\sin\theta$ for small amplitude α.

e. Verify that $\theta(t)=\alpha\cos\left(\sqrt{\frac{g}{L}}\,t\right)$ is a solution to $\frac{d^2\theta}{dt^2}=-\frac{g}{L}\theta$.

Hence find the period of the oscillation for small amplitude α.

Now investigate the period of oscillation for large α.

f. From part a, $\frac{d\theta}{dt}=-\sqrt{\frac{2g}{L}(\cos\theta-\cos\alpha)}$, find $\frac{dt}{d\theta}$ as a function of θ in terms of g, L and α.

g. Show that the time taken by the object to move from $\theta=\phi$ to $\theta=0$ is given by the definite integral $t=\sqrt{\frac{L}{2g}}\int_0^{\phi}\frac{1}{\sqrt{\cos\theta-\cos\alpha}}\,d\theta$, where $0<\phi<\alpha$.

The period of the oscillation is given by $T = 4\sqrt{\dfrac{L}{2g}} \int_0^{\alpha} \dfrac{1}{\sqrt{\cos\theta - \cos\alpha}}\, d\theta$.

Let $L = 1$ for part h and part i.

h. Use the definite integral to calculate T for small $\alpha = 0.08$ and $\alpha = 0.09$, and for large $\alpha = 1.54$ and $\alpha = 1.55$. Correct your answers to 3 decimal places.
Briefly comment on the **change** in the period T of the oscillation when α increases from small to large values.
Further justify your answer by considering $\dfrac{dT}{d\alpha}$.

The definite integral $T = 4\sqrt{\dfrac{L}{2g}} \int_0^{\alpha} \dfrac{1}{\sqrt{\cos\theta - \cos\alpha}}\, d\theta$ can be expressed as an **infinite series**.

Only the first **five** terms of the infinite series are shown below.

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{\alpha}{2} + \left(\frac{1\times 3}{2\times 4}\right)^2 \sin^4\frac{\alpha}{2} + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2 \sin^6\frac{\alpha}{2} + \left(\frac{1\times 3\times 5\times 7}{2\times 4\times 6\times 8}\right)^2 \sin^8\frac{\alpha}{2} + \cdots\cdots\right)$$

i. Calculate the approximate value of T, *using only the first* **four** t*erms of the infinite series*, for each of $\alpha = 0.09$ and $\alpha = 1.55$.

End of Part II End of Problem Solving Task

